STRONG CONVERGENCE OF A UNIFORM KERNEL ESTIMATOR FOR INTENSITY OF A PERIODIC POISSON PROCESS WITH UNKNOWN PERIOD

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ABSTRACT. Strong convergence of a uniform kernel estimator for intensity of a periodic Poisson process with unknowm period is presented and proved. The result presented here is a special case of the one in [3]. The aim of this paper is to present an alternative and a relatively simpler proof of strong convergence compared to the one in [3]. This is a joint work with R. Helmers and R. Zitikis.

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1. Introduction and main result

In this paper, strong convergence of a uniform kernel estimator for intensity of a periodic Poisson process with unknowm period is presented and proved. For more general results which using general kernel function can be found in [3] and chapter 3 of [4].

Let X be a Poisson process on $[0, \infty)$ with (unknown) locally integrable intensity function λ. We assume that λ is a periodic function with unknown period τ. We do not assume any parametric form of λ, except that it is periodic. That is, for each point $s \in [0, \infty)$ and all $k \in \mathbf{Z}$, with $\mathbf{Z}$ denotes the set of integers, we have

$$\lambda(s + k\tau) = \lambda(s). \tag{1.1}$$

Suppose that, for some $\omega \in \Omega$, a single realization $X(\omega)$ of the Poisson process X defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with intensity function λ is observed, though only within a bounded interval $[0, n]$. Our goal is: (a) To present a uniform kernel estimator for λ at a given point $s \in [0, n]$ using only a single realization $X(\omega)$ of the Poisson process X observed in interval $[0, n]$. (The requirement $s \in [0, n]$ can be dropped if we know the period τ.) (b) To determine an alternative set of conditions for having strong convergence of this estimator compared to the one in [3]. (c) To present an alternative and a relatively simpler proof of strong convergence of the estimator compared to the one in [3].

Note that, since λ is a periodic function with period τ, the problem of estimating λ at a given point $s \in [0, n]$ can be reduced into a problem of estimating λ at a given point $s \in [0, \tau)$. Hence, for the rest of this paper, we assume that $s \in [0, \tau)$.

We will assume throughout that s is a Lebesgue point of λ, that is we have

$$\lim_{h \downarrow 0} \frac{1}{2h} \int_{-h}^{h} |\lambda(s+x) - \lambda(s)| dx = 0$$

(e.g. [7], p.107-108). This assumption is a mild one since the set of all Lebesgue points of λ is dense in $\mathbf{R}$, whenever λ is assumed to be locally integrable.

Let $\hat{\tau}_n$ be any consistent estimator of the period τ, that is,

$$\hat{\tau}_n \xrightarrow{p} \tau,$$

as $n \to \infty$. For example, one may use the estimators constructed in [2] or perhaps the estimator investigated by [6] or [1]. Let also h_n be a sequence of positive real numbers converging to 0, that is,

$$h_n \downarrow 0 \tag{1.2}$$

as $n \to \infty$. With these notations, we may define an estimator of $\lambda(s)$ as

$$\hat{\lambda}_n(s) := \frac{\hat{\tau}_n}{n} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} X\left([s + k\hat{\tau}_n - h_n, s + k\hat{\tau}_n - h_n] \cap [0, n]\right). \tag{1.3}$$

The idea behind the construction of the estimator $\hat{\lambda}_n(s)$ given in (1.3) can be found e.g. in [5].

The main result of this paper is the following theorem.

Theorem 1.1. *Let the intensity function λ be periodic and locally integrable. Furthermore, let the bandwidth h_n be such that (1.2) holds true, and*

$$\frac{1}{nh_n} = \mathcal{O}(n^{-\alpha}) \tag{1.4}$$

and

$$n|\hat{\tau}_n - \tau|/h_n = \mathcal{O}(n^{-\beta}) \tag{1.5}$$

with probability 1, as $n \to \infty$, for an arbitrarily small $\alpha > 0$ and $\beta > 0$, then

$$\hat{\lambda}_n(s) \stackrel{a.s.}{\to} \lambda(s) \tag{1.6}$$

as $n \to \infty$, provided s is a Lebesgue point of λ. In other words, $\hat{\lambda}_n(s)$ converges strongly to $\lambda(s)$ as $n \to \infty$.

2. Proofs of Theorem 1.1

Throughout this paper, for any random variables Y_n and Y on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$, we write $Y_n \stackrel{c}{\to} Y$ to denote that Y_n *converges completely* to Y, as $n \to \infty$. We say that Y_n converges completely to Y if

$$\sum_{n=1}^{\infty} \mathbf{P}(|Y_n - Y| > \epsilon) < \infty,$$

for every $\epsilon > 0$.

Let $B_h(x)$ denotes the interval $[x-h, x+h]$. To establish Theorem 1.1, first we prove

$$\frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} X\left(B_{h_n}(s + k\hat{\tau}_n) \cap [0,n]\right) \stackrel{a.s.}{\to} \lambda(s), \tag{2.1}$$

as $n \to \infty$, where $N_n = \#\{k : \ s + k\tau \in [0,n]\}$. To prove (2.1), by Borel-Cantelli, it suffices to check, for each $\epsilon > 0$, that

$$\sum_{n=1}^{\infty} \mathbf{P}\left(\left| \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} X\left(B_{h_n}(s + k\hat{\tau}_n) \cap [0,n]\right) - \lambda(s) \right| > \epsilon \right) < \infty, \tag{2.2}$$

i.e. the difference between the quantity on the l.h.s. of (2.1) and $\lambda(s)$ converges completely to zero, as $n \to \infty$. By Lemma 2.1, Lemma 2.2, and Lemma 2.3, we obtain (2.2), which implies (2.1).

Then, to prove (1.6), it remains to check that $\hat{\lambda}_n(s)$ can be replaced by the quantity on the l.h.s. of (2.1), i.e. we must show that the difference between $\hat{\lambda}_n(s)$ and the quantity on the l.h.s. of (2.1) converges almost surely to zero, as $n \to \infty$. To show this, first we write this difference as

$$\left(\frac{\hat{\tau}_n N_n}{n} - 1 \right) \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} X\left(B_{h_n}(s + k\hat{\tau}_n) \cap [0,n]\right), \tag{2.3}$$

that is, the quantity on the l.h.s. of (2.1) multiplied by $(\hat{\tau}_n N_n n^{-1} - 1)$. Since $\lambda(s)$ is finite, by (2.1), we have that the quantity on the l.h.s.

of (2.1) is $\mathcal{O}(1)$, with probability 1, as $n \to \infty$. Hence, it remains to check that

$$\left|\frac{\hat{\tau}_n N_n}{n} - 1\right| = o(1), \tag{2.4}$$

with probability 1, as $n \to \infty$. By the triangle inequality, the quantity on the l.h.s. of (2.4) does not exceed

$$\left|\frac{\hat{\tau}_n N_n}{n} - \frac{\hat{\tau}_n}{\tau}\right| + \left|\frac{\hat{\tau}_n}{\tau} - 1\right| \leq \frac{\hat{\tau}_n}{n}\left|N_n - \frac{n}{\tau}\right| + \frac{1}{\tau}\left|\hat{\tau}_n - \tau\right|. \tag{2.5}$$

Note that $|n/\tau - N_n| \leq 1$, and $\hat{\tau}_n = \mathcal{O}(1)$, with probability 1, as $n \to \infty$ (by (1.5)). Hence, the first term on the r.h.s. of (2.5) is $\mathcal{O}(n^{-1})$, with probability 1, as $n \to \infty$. By (1.5), we have that its second term is $o(1)$, with probability 1, as $n \to \infty$. Therefore we have (2.4). This completes the proof of Theorem 1.1.

In the following lemma we shall show that we may replace the random centre $s + k\hat{\tau}_n$ of the interval $B_{h_n}(s + k\hat{\tau}_n)$ in (2.1) by its deterministic limit $s + k\tau$.

Lemma 2.1. *Suppose λ is periodic (with period τ) and locally integrable. If, in addition, (1.2) and (1.5) are satisfied, then*

$$\frac{1}{N_n}\sum_{k=-\infty}^{\infty}\frac{1}{2h_n}\left|\{X\left(B_{h_n}(s+k\hat{\tau}_n)\cap[0,n]\right) - X\left(B_{h_n}(s+k\tau)\cap[0,n]\right)\}\right|$$
$$\stackrel{c}{\to} 0, \tag{2.6}$$

as $n \to \infty$, provided s is a Lebesgue point of λ.

Proof: First note that the difference within curly brackets on the l.h.s. of (2.6) does not exceed

$$X\left(B_{h_n}(s+k\hat{\tau}_n)\Delta B_{h_n}(s+k\tau)\cap[0,n]\right). \tag{2.7}$$

Now we notice that

$$B_{h_n - |k(\hat{\tau}_n - \tau)|}(s+k\tau) \subseteq B_{h_n}(s+k\hat{\tau}_n) \subseteq B_{h_n + |k(\hat{\tau}_n - \tau)|}(s+k\tau). \tag{2.8}$$

By (2.7) and (2.8) we have

$$\begin{aligned}&\left|\{X\left(B_{h_n}(s+k\hat{\tau}_n)\cap[0,n]\right) - X\left(B_{h_n}(s+k\tau)\cap[0,n]\right)\}\right|\\ &\leq 2X\left(B_{h_n+|k(\hat{\tau}_n-\tau)|}(s+k\tau)\setminus B_{h_n-|k(\hat{\tau}_n-\tau)|}(s+k\tau)\cap[0,n]\right).\end{aligned} \tag{2.9}$$

Hence, to prove (2.6), it suffices to show that

$$\frac{1}{N_n}\sum_{k=-\infty}^{\infty}\frac{1}{h_n}X\left(B_{h_n+|k(\hat{\tau}_n-\tau)|}(s+k\tau)\setminus B_{h_n-|k(\hat{\tau}_n-\tau)|}(s+k\tau)\cap[0,n]\right)$$
$$\stackrel{c}{\to} 0, \tag{2.10}$$

as $n\to\infty$. To prove (2.10) we argue as follows. Let Λ_n denotes the l.h.s. of (2.10), and let also $\epsilon>0$ be any fixed real number. Then to verify (2.10) it suffices to check, for each $\epsilon>0$,

$$\sum_{n=1}^{\infty}\mathbf{P}\left(|\Lambda_n|>\epsilon\right)<\infty. \tag{2.11}$$

By the assumption (1.5), there exists large fixed positive integer n_0 and posistive constant C such that $n|\hat{\tau}_n-\tau|\le Cn^{-\beta}h_n$ with probability 1, for all $n\ge n_0$. Then, for all $n\ge n_0$, we have with probability 1 that $\mathbf{P}(|\Lambda_n|>\epsilon)\le\mathbf{P}(|\bar{\bar{\Lambda}}_n|>\epsilon)$, where $\bar{\bar{\Lambda}}_n$ is given by

$$\bar{\bar{\Lambda}}_n = \frac{1}{N_n}\sum_{k=-\infty}^{\infty}\frac{1}{2h_n}$$
$$X\left(B_{h_n(1+Cn^{-\beta})}(s+k\tau)\setminus B_{h_n(1-Cn^{-\beta})}(s+k\tau)\cap[0,n]\right). \tag{2.12}$$

(Note that $\bar{\bar{\Lambda}}_n$ is precisely equal to $\bar{\Lambda}_n$ in (2.10), provided we replace, for our present purposes, δ by $Cn^{-\beta}$). Since to show convergency of an infinite series it suffices to check convergency of its tail, to prove (2.11), it suffices to check, for each $\epsilon>0$, that

$$\sum_{n=n_0}^{\infty}\mathbf{P}\left(|\bar{\bar{\Lambda}}_n|>\epsilon\right)<\infty. \tag{2.13}$$

By Markov inequality for the M-th moment, we then obtain

$$\mathbf{P}\left(|\bar{\bar{\Lambda}}_n|>\epsilon\right)\le\frac{E(\bar{\bar{\Lambda}}_n)^M}{\epsilon^M}=\left(\frac{1}{2\epsilon N_n h_n}\right)^M$$
$$\mathbf{E}\left(\sum_{k=-\infty}^{\infty}X\left(B_{h_n(1+Cn^{-\beta})}(s+k\tau)\setminus B_{h_n(1-Cn^{-\beta})}(s+k\tau)\cap W_n\right)\right)^M. \tag{2.14}$$

Now consider the expectation on the r.h.s. of (2.14). By writing the M-th power of a sum as a M-multiple sum, we can interchange summations and expectation. Note that for large n, by (1.2), the random variables

$X\left(B_{h_n(1+Cn^{-\beta})}(s+k\tau)\setminus B_{h_n(1-Cn^{-\beta})}(s+k\tau)\right)$ and

$$X\left(B_{h_n(1+Cn^{-\beta})}(s+j\tau)\setminus B_{h_n(1-Cn^{-\beta})}(s+j\tau)\right)$$

for $k \neq j$, are independent. Now, we distinguish M different cases in the M-multiple sum, namely, case (1) if all indexes are the same, up to case (M) if all indexes are different. Then we split up the M-multiple sum into M different terms, where each term corresponds to each of the M cases. Because for each $k \in \mathbb{Z}$ and for any fixed M, by (1.2), it is easy to check that

$$\mathbf{E}\left(X\left(B_{h_n(1+Cn^{-\beta})}(s+k\tau)\setminus B_{h_n(1-Cn^{-\beta})}(s+k\tau)\right)\right)^M = O(1), \quad (2.15)$$

as $n \to \infty$, uniformly in k, we find that for large n, the biggest term among those M terms, is the term corresponds to the case where all indexes are different. Hence we conclude that the expectation on the r.h.s. of (2.14) does not exceed

$$\begin{aligned} &M\left(\sum_{k=-\infty}^{\infty}\mathbf{E}X\left(B_{h_n(1+Cn^{-\beta})}(s+k\tau)\setminus B_{h_n(1-Cn^{-\beta})}(s+k\tau)\cap W_n\right)\right)^M \\ &= M\left(\int_{B_{(1+Cn^{-\beta})h_n}(0)\setminus B_{(1-Cn^{-\beta})h_n}(0)}\lambda(s+x)\right. \\ &\quad\left.\sum_{k=-\infty}^{\infty}\mathbf{I}(s+k\tau+x\in W_n)dx\right)^M \\ &\leq M\left(N_n+1\right)^M\left(\int_{B_{(1+Cn^{-\beta})h_n}(0)\setminus B_{(1-Cn^{-\beta})h_n}(0)}\lambda(s+x)dx\right)^M. \end{aligned} \quad (2.16)$$

The integral on the r.h.s. of (2.16) does not exceed

$$\begin{aligned} &\int_{B_{(1+Cn^{-\beta})h_n}(0)\setminus B_{(1-Cn^{-\beta})h_n}(0)}|\lambda(s+x)-\lambda(s)|dx \\ &+\left|B_{(1+Cn^{-\beta})h_n}(0)\setminus B_{(1-Cn^{-\beta})h_n}(0)\right|\lambda(s). \end{aligned} \quad (2.17)$$

Since s is a Lebesgue point of λ, we have that the quantity in the first term of (2.17) is of order $o(n^{-\beta}h_n)$, as $n \to \infty$. Since $\lambda(s)$ is finite and $\left|B_{(1+Cn^{-\beta})h_n}(0)\setminus B_{(1-Cn^{-\beta})h_n}(0)\right| = 4Cn^{-\beta}h_n$, we have that the quantity in the second term of (2.17) is of order $O(n^{-\beta}h_n)$, as $n \to \infty$. Hence, the r.h.s. of (2.16) is of order $O\left(n^{M(1-\beta)}h_n^M\right)$, which implies that the r.h.s. of (2.14) is of order $O\left(n^{-M\beta}\right)$, as $n \to \infty$. By choosing $M > \frac{1}{\beta}$, we see that (2.13) is proved. This completes the proof of Lemma 2.1.

To complete our proof of Theorem 1.1 we also need the following lemma.

Lemma 2.2. *Suppose λ is periodic (with period τ) and locally integrable. If, in addition, (1.2) and (1.4) are satisfied, then*

$$\frac{1}{N_n}\sum_{k=-\infty}^{\infty}\frac{1}{2h_n}\left|X\left(B_{h_n}(s+k\tau)\cap[0,n]\right)-\mathbf{E}X\left(B_{h_n}(s+k\tau)\cap[0,n]\right)\right|$$
$$\xrightarrow{c} 0, \qquad (2.18)$$

as $n\to\infty$, provided s is a Lebesgue point of λ.

Proof: First we write the l.h.s. of (2.18) as

$$\frac{1}{2N_nh_n}\left|\sum_{k=-\infty}^{\infty}\tilde{X}\left(B_{h_n}(s+k\tau)\cap W_n\right)\right|, \qquad (2.19)$$

where we write $\tilde{X}$ to denote $X-\mathbf{E}X$. By Markov inequality for the $2M$-th moment, for each $\epsilon>0$, we then obtain

$$\mathbf{P}\left(\frac{1}{2N_nh_n}\left|\sum_{k=-\infty}^{\infty}\tilde{X}\left(B_{h_n}(s+k\tau)\cap W_n\right)\right|>\epsilon\right)$$
$$\leq\left(\frac{1}{2\epsilon N_nh_n}\right)^{2M}\mathbf{E}\left(\sum_{k=-\infty}^{\infty}\tilde{X}\left(B_{h_n}(s+k\tau)\cap W_n\right)\right)^{2M}. \qquad (2.20)$$

Now consider the expectation on the r.h.s. of (2.20). By writing the $2M$-th power of a sum as a $2M$-multiple sum, we can interchange summations and expectation. For large n, the r.v. $X\left(B_{h_n}(s+k\tau)\cap W_n\right)$ and $X\left(B_{h_n}(s+j\tau)\cap W_n\right)$, for $k\neq j$, are independent. Here we also distinguish $2M$ different cases in the $2M$-multiple sum, namely, case (1) if all indexes are the same, up to case $(2M)$ if all indexes are different. Then we also split up the $2M$-multiple sum into $2M$ different terms, where each term corresponds to each of the $2M$ cases. Because for any fixed M, it is easy to check that $\mathbf{E}\tilde{X}\left(B_{h_n}(s+k\tau)\cap W_n\right)=0$ and $\mathbf{E}\left(\tilde{X}\left(B_{h_n}(s+k\tau)\cap W_n\right)\right)^{2M}=O(1)$ as $n\to\infty$, uniformly in k, we find for large n, the biggest term among those $2M$ terms, is the one corresponds to the case where there are M pairs of the same indexes. Hence we conclude that the expectation on the r.h.s. of (2.20) does not exceed

$$\begin{aligned}
&2M\left(\sum_{k=-\infty}^{\infty}\mathbf{E}\left(\tilde{X}\left(B_{h_n}(s+k\tau)\cap W_n\right)\right)^2\right)^M\\
&= M2^{M+1}h_n^M\left(\sum_{k=-\infty}^{\infty}\frac{1}{2h_n}\int_{B_{h_n}(0)}\lambda(s+x)\mathbf{I}\left(s+k\tau+x\in W_n\right)dx\right)^M\\
&\le M2^{M+1}h_n^M(N_n+1)^M\left(\frac{1}{2h_n}\int_{B_{h_n}(0)}\lambda(s+x)dx\right)^M\\
&= O(n^M h_n^M), \qquad (2.21)
\end{aligned}$$

as $n\to\infty$. Combining this result with the assumption (1.4), we then obtain that the r.h.s. of (2.20) is of order $O\left(n^{-M}h_n^{-M}\right)=O\left(n^{-M\alpha}\right)$, as $n\to\infty$. By choosing $M>\frac{1}{\alpha}$, we have that the probabilities on the l.h.s. of (2.20) are summable, which implies this lemma. This completes the proof of Lemma 2.2.

It remains to evaluate a non-random sum.

Lemma 2.3. *Suppose λ is periodic (with period τ) and locally integrable. If, in addition, (1.2) is satisfied, then*

$$\frac{1}{N_n}\sum_{k=-\infty}^{\infty}\frac{1}{2h_n}\mathbf{E}X\left(B_{h_n}(s+k\tau)\cap[0,n]\right)=\lambda(s)+o(1), \qquad (2.22)$$

as $n\to\infty$, provided s is a Lebesgue point of λ.

Proof: Using the fact that X is Poisson, the l.h.s. of (2.22) can be written as

$$\begin{aligned}
&\frac{1}{N_n}\sum_{k=-\infty}^{\infty}\frac{1}{2h_n}\int_{-h_n}^{h_n}\lambda(s+k\tau+x)\mathbf{I}(s+k\tau+x\in[0,n])dx\\
&=\frac{1}{2N_nh_n}\int_{-h_n}^{h_n}\lambda(s+x)\sum_{k=-\infty}^{\infty}\mathbf{I}(s+k\tau+x\in[0,n])dx. \qquad (2.23)
\end{aligned}$$

Now note that

$$(N_n-1)\le\sum_{k=-\infty}^{\infty}\mathbf{I}(s+k\tau+x\in[0,n])\le(N_n+1),$$

which implies $N_n^{-1}\sum_{k=-\infty}^{\infty}\mathbf{I}(s+k\tau+x\in[0,n])$ can be written as $(1+\mathcal{O}(n^{-1}))$, as $n\to\infty$, uniformly in x. Then, the quantity on the r.h.s. of (2.23) can be written as

$$\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)\frac{1}{2h_n}\int_{-h_n}^{h_n}\lambda(s+x)dx. \qquad (2.24)$$

By (1.2) together with the assumption that s is a Lebesgue point of λ, we have that

$$(2h_n)^{-1}\int_{-h_n}^{h_n}\lambda(s+x)dx=\lambda(s)+o(1),$$

as $n\to\infty$. Then we obtain this lemma. This completes the proof of Lemma 2.3.

Lemma 2.4. *Suppose that the assumption (1.5) is satisfied. Then, for each positive integer M, we have that*

$$\mathbf{E}\left(\hat{\tau}_n-\tau\right)^{2M}=O\left(n^{-2M(1+\beta)}h_n^{2M}\right), \tag{2.25}$$

as $n\to\infty$.

Proof: By the assumption (1.5), there exists large positive constant C and positive integer n_0 such that

$$|\hat{\tau}_n-\tau|\leq Cn^{-(1+\beta)}h_n, \tag{2.26}$$

with probability 1, for all $n\geq n_0$. Then, the l.h.s. of (2.25) can be written as

$$\begin{aligned}&\int_0^\infty x^{2M}d\mathbf{P}\left(|\hat{\tau}_n-\tau|\leq x\right)\\&=-\int_0^{Cn^{-(1+\beta)}h_n}x^{2M}d\mathbf{P}\left(|\hat{\tau}_n-\tau|>x\right).\end{aligned} \tag{2.27}$$

By partial integration, the r.h.s. of (2.27) is equal to

$$\begin{aligned}&-x^{2M}\mathbf{P}\left(|\hat{\tau}_n-\tau|>x\right)|_0^{Cn^{-(1+\beta)}h_n}\\&+2M\int_0^{Cn^{-(1+\beta)}h_n}\mathbf{P}\left(|\hat{\tau}_n-\tau|>x\right)x^{2M-1}dx.\end{aligned} \tag{2.28}$$

The first term of (2.28) is equals to zero, while its second term is at most equal to

$$\begin{aligned}2M\int_0^{Cn^{-(1+\beta)}h_n}x^{2M-1}dx&=C^{2M}n^{-2M(1+\beta)}h_n^{2M}\\&=O\left(n^{-2M(1+\beta)}h_n^{2M}\right),\end{aligned} \tag{2.29}$$

as $n\to\infty$. This completes the proof of Lemma 2.4.

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