REGULATION LIMITS UNDER CONTROL EFFORT OF SIMO LTI SYSTEMS AND ITS EXTENSION TO DELAY-TIME SYSTEMS

TONI BAKHTIAR

Departemen of Mathematics,
Faculty of Mathematics and Natural Sciences,
Bogor Agricultural University
Darmaga, Bogor 16680, Indonesia

Abstract. In this paper, we investigate regulation properties pertaining to SIMO LTI systems, in which objective function of regulated response is minimized jointly with the control effort. We provide the closed-form solution of the $\mathcal{H}_2$ optimal regulation performance for unstable/non-minimum phase continuous-time and discrete-time systems. A direct implication of our main result includes the energy regulation performance of minimum phase time delay systems.
Keywords: Performance limitations, $\mathcal{H}_2$ optimal control, SIMO systems, unstable/non-minimum phase systems.

1. Introduction

The study on control performance limitations achievable by feedback control systems is one of the important research topics in control theory, and it has been paid much attention in the recent years [1]–[6]. In this study certain classical optimal problems are examined under optimality criteria formulated in time or frequency domain, which have led to closed-form solutions of the best achievable performance. One of such well-studied problems is the optimal regulation problem.

The optimal regulation performance is measured by minimizing the energy of control input, or by minimizing the energy of control input jointly with the energy of system output. We call the former the energy regulation problem and the latter the output regulation problem. Results on $\mathcal{H}_2$ energy regulation problem can be found in [5] for continuous-time system and in [4] for discrete-time system. Both results are conducted for unstable/non-minimum phase SISO/SIMO plants. Equivalent results in minimum phase SISO systems but articulated in term of signal-to-noise ratio constrained channels are in [1]. Meanwhile, result on $\mathcal{H}_2$ output regulation problem is presented in [3] for unstable/minimum phase SISO/MIMO continuous-time systems.

This paper discusses the output regulation problem of unstable/non-minimum phase SIMO continuous-time and discrete-time systems. An

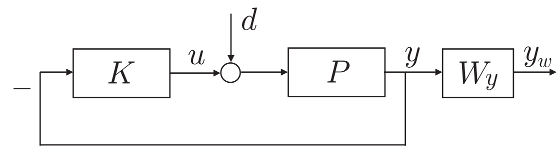

Figure 1. The regulation scheme.

implication relates to energy regulation problem of time delay systems is provided.

The rest of this paper is organized as follows. In Section 2 we describe the notation and state some preliminaries. Section 3 provides a closed-form solution of the optimal regulation performance for continuous-time systems, and that of discrete-time systems is given in Section 4. An implication of the main result to time delay systems is given in Section 5. Some concluding statements are in Section 6.

2. Preliminaries

We give a brief description of the notation used throughout this report. We denote the real set by $\mathbb{R}$ and the complex set by $\mathbb{C}$. For any $c \in \mathbb{C}$, its complex conjugate is denoted by $\bar{c}$. For any vector u we shall use u^T, u^H, and $\|u\|$ as its transpose, conjugate transpose, and Euclidean norm, respectively. We call the one-dimensional subspace spanned by u the direction of u. For any matrix $A \in \mathbb{C}^{m\times n}$, we denote its conjugate transpose by A^H and its column space by $\mathbb{R}[A]$. The cardinality of a set S is denoted by $\#S$. In s-domain analysis, i.e., continuous-time case, let the open left half plane be denoted by $\mathbb{C}_- := \{s \in \mathbb{C} : \operatorname{Re} s < 0\}$, the open right half plane by $\mathbb{C}_+ := \{s \in \mathbb{C} : \operatorname{Re} s > 0\}$, and the imaginary axis by $\mathbb{C}_0$. And for any matrix function $f \in \mathbb{C}^{m\times n}$ we define $f^\sim(s) := f^T(-s)$. For any signal $x(t)$, $t > 0$, we define its Laplace transform $\hat{x}(s)$ by

$$\hat{x}(s) = \mathcal{L}\{x(t)\} := \int_0^\infty x(t)e^{-st}\,dt.$$

While in z-domain analysis, i.e., discrete-time case, the unit circle is denoted by $\partial\mathbb{D} := \{z \in \mathbb{C} : |z| = 1\}$. We also define the following sets: $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{D}^c := \{z \in \mathbb{C} : |z| \geq 1\}$, and $\mathbb{D}^c := \{z \in \mathbb{C} : |z| > 1\}$. Clearly, $\mathbb{D}$ and $\mathbb{D}^c$ respectively can be seen as the regions inside and outside unit circle. Furthermore, we define $f^\sim(z) := f^T(z^{-1})$. For any sequence $x(k)$, $k = 0, 1, \ldots$, we define its $\mathcal{Z}$ transform $\hat{x}(z)$ by

$$\hat{x}(z) = \mathcal{Z}\{x(k)\} := \sum_{k=0}^{\infty} x(k)z^{-k}.$$

The standard setup under consideration in this paper is the SIMO feedback system depicted in Fig. 1, where P represents the plant, K the stabilizing compensator, and W_y the stable/minimum phase weighting function. The signals $d \in \mathbb{R}$, $u \in \mathbb{R}$, $y \in \mathbb{R}^m$, and $y_w \in \mathbb{R}^m$ are the disturbance input, the plant input, the system output, and the weighted system output, respectively.

For the plant rational transfer function P, its left and right coprime factorization be given by

$$P = NM^{-1} = \tilde{M}^{-1}\tilde{N}, \tag{1}$$

where $N, M, \tilde{N}, \tilde{M} \in \mathbb{R}\mathcal{H}_\infty$ and they satisfy the double Bezout identity

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ N & X \end{bmatrix} = I \tag{2}$$

for some $X, Y, \tilde{X}, \tilde{Y} \in \mathbb{R}\mathcal{H}_\infty$. All the stabilizing compensators K can be characterized by

$$\begin{aligned} \mathcal{K} \;:=\; & \{K : K = (Y - MQ)(NQ - X)^{-1} \\ & = (Q\tilde{N} - \tilde{X})^{-1}(\tilde{Y} - Q\tilde{M});\ Q \in \mathbb{R}\mathcal{H}_\infty\}. \end{aligned} \tag{3}$$

A complex number z is said to be a zero of P if $P_i(z) = 0$. In addition, if z lies either in $\mathbb{C}_+$ for s-domain or $\mathbb{D}^c$ for z-domain then z is said to be a non-minimum phase zero. P is said to be minimum phase if it has no non-minimum phase zero; otherwise, it is said to be non-minimum phase. On the other hand, a complex number p is said to be a pole of P if $P(p)$ is unbounded. A pole p is said to be unstable if it lies in $\mathbb{C}_+$ or $\mathbb{D}^c$. P is said to be stable if it has no unstable pole; otherwise, unstable. For technical reasons, it is assumed that the plant does not have zeros and poles at the same location.

A transfer function N, not necessarily square, is called an inner if N is in $\mathbb{R}\mathcal{H}_\infty$ and $N^\sim N = I$ for all $s = \mathrm{j}\omega$ or $z = e^{\mathrm{j}\theta}$ and is called co-inner if $N \in \mathbb{R}\mathcal{H}_\infty$ and $NN^\sim = I$. A transfer function M is called outer if M is in $\mathbb{R}\mathcal{H}_\infty$ and has a right inverse which is analytic in $\mathbb{C}_+$ or $\mathbb{D}^c$. For an arbitrary $P \in \mathbb{R}\mathcal{H}_\infty$,

$$P = \Lambda_i \Lambda_o, \tag{4}$$

where Λ_i is inner and Λ_o is outer, is defined as an inner-outer factorization of P. We call Λ_i the inner factor and Λ_o the outer factor.

In subsequent analysis, we let P and K be

$$P = [P_1, P_2, \ldots, P_m]^T, \tag{5}$$

$$K = [K_1, K_2, \ldots, K_m], \tag{6}$$

with P_i and K_i, $i = 1, \ldots, m$, are scalar transfer functions. In the present work, we consider an impulse function as the disturbance signal d.

3. Continuous-time Systems

Suppose that the plant $P(s)$ is given in (5) and its coprime factorization is given in (1). Without loss of generality we may set $M = B$, where

$$B(s) = \prod_{i=1}^{N_\lambda} \frac{s - \lambda_i}{s + \bar{\lambda}_i} \tag{7}$$

We denote by λ_i, $i = 1, \ldots, N_\lambda$, the unstable poles of $P(s)$. It is useful to point out that $B(\infty) = 1$.

We minimize the performance index

$$E_c := \int_0^\infty \left(\|y_w(t)\|^2 + |u(t)|^2 \right) dt, \tag{8}$$

where $y_w(t)$ is the weighted system output, i.e.,

$$y_w(t) = \mathcal{L}^{-1}\{W_y(s)\hat{y}(s)\}.$$

In order for E_c to be finite, it is necessary that $P\hat{d} \in \mathcal{L}_2$, where $\mathcal{L}_2$ is a Hilbert space with an inner product

$$\langle f_1, f_2 \rangle := \frac{1}{2\pi} \int_{-\pi}^{\pi} f_1^H(e^{j\theta}) f_2(e^{j\theta})\, d\theta. \tag{9}$$

Since $d(t)$ is an impulse function so that $\hat{d}(s) = 1$, then we need the following assumption.

Assumption 1. *$P(s)$ is strictly proper, i.e., $P(\infty) = 0$.*

This assumption implies that $y(0)$ is finite, a necessary condition for the output energy to be finite.

Theorem 1. *Suppose that the plant $P(s)$ has unstable poles λ_i, $i = 1, \ldots, N_\lambda$ and its coprime factorization is given by (1). Let define the inner-outer factorization*

$$\begin{bmatrix} W_y N \\ -1 \end{bmatrix} = \Lambda_i \Lambda_o.$$

Then,

$$E_c^* = E_1 + E_2, \tag{10}$$

where

$$\begin{aligned} E_1 &= 2\sum_{i=1}^{N_\lambda} \lambda_i + \frac{1}{\pi}\int_0^\infty \log\left(1 + \|W_y(j\omega)P(j\omega)\|^2\right) d\omega \\ E_2 &= \sum_{i,j\in\mathbb{N}} \frac{4Re(z_i)Re(z_j)}{\bar{b}_i b_j(\bar{z}_i + z_j)} (1 - \Lambda_o(z_i)B^{-1}(z_i))^H (1 - \Lambda_o(z_j)B^{-1}(z_j)), \end{aligned}$$

with

$$b_i = \prod_{j\in\mathbb{N}, j\neq i} \frac{z_j - z_i}{z_j + \bar{z}_i},$$

and $\mathbb{N} := \{i : \tilde{N}(z_i) = 0,\ z_i \in \mathbb{C}_+\}$.

Proof: The proof of E_1 can be found in [3]. To prove E_2, follow the way in [5]. □

Theorem 1 shows that the regulation performance depends not only on the plant unstable poles and non-minimum phase zeros but also on its gain and a certain of outer factor. When $W_y = 0$, we have the following result which is consistent with an existing result in [5].

Corollary 1. *If $W_y = 0$ which implies $\Lambda_o = 1$, then*

$$E_c^* = 2\sum_{i=1}^{N_\lambda} \lambda_i + \sum_{i,j\in\mathbb{N}} \frac{4Re(z_i)Re(z_j)}{\bar{b}_i b_j(\bar{z}_i + z_j)}(1 - B^{-1}(z_i))^H(1 - B^{-1}(z_j)).$$

4. Discrete-time Systems

In (1), it is possible to set $M = B$, where

$$B(z) = \prod_{i=1}^{N_p} \frac{z - p_i}{\bar{p}_i z - 1}, \tag{11}$$

with p_i, $i = 1, \ldots, N_p$, the unstable poles of $P(z)$. Note that $B(\infty) = \prod_{i=1}^{N_p} \frac{1}{\bar{p}_i}$, and $\hat{d}(z) = 1$. Also note that in regulation problem of discrete-time systems, we do not need a kind of Assumption 1. We minimize the performance index

$$E_d := \sum_{k=0}^{\infty} \left(\|y_w(k)\|^2 + |u(k)|^2\right), \tag{12}$$

where $y_w(k)$ is the weighted system output, i.e.,

$$y_w(k) = \mathcal{Z}^{-1}\{W_y(z)\hat{y}(z)\}.$$

Lemma 1. *If f is scalar transfer function and $f(z) \in \mathbb{R}\mathcal{H}_\infty$, then*

$$\frac{1}{\pi}\int_{-\pi}^{\pi} Re\{f(e^{j\theta})\}\, d\theta = 2f(\infty). \tag{13}$$

Theorem 2. *Suppose that $P(z)$ has unstable poles p_i, $i = 1, \ldots, N_p$ and its coprime factorization is given by (1). Let define the inner-outer factorization*

$$\begin{bmatrix} W_y N \\ -1 \end{bmatrix} = \Lambda_i \Lambda_o.$$

Then,

$$E_d^* = E_1 + E_2, \tag{14}$$

where

$$\begin{aligned} E_1 &= |\Lambda_o(\infty)|^2 \prod_{i=1}^{N_p} |p_i|^2 - 1, \\ E_2 &= \sum_{i,j\in\mathbb{N}} \frac{(|s_i|^2-1)(|s_j|^2-1)}{\bar{b}_i b_j(\bar{s}_i s_j - 1)} \times \\ & \quad \left(\Lambda_o(\infty)B^{-1}(\infty) - \Lambda_o(s_i)B^{-1}(s_i)\right)^H \times \\ & \quad \left(\Lambda_o(\infty)B^{-1}(\infty) - \Lambda_o(s_j)B^{-1}(s_j)\right), \end{aligned}$$

with

$$b_i = \prod_{j\in\mathbb{N}, j\neq i} \frac{s_i - s_j}{s_i \bar{s}_j - 1},$$

and $\mathbb{N} := \{i : \tilde{N}(s_i) = 0, s_i \in \bar{\mathbb{D}}^c\}$.

Proof: From (1)–(3) we may express (12) as

$$E_d = \left\| \begin{bmatrix} W_y(X\tilde{N} - NQ\tilde{N}) \\ B^{-1}Y\tilde{N} - Q\tilde{N} \end{bmatrix} \right\|_2^2.$$

After a lengthy manipulation, we then can show that $E_d^* = \tilde{E}_1 + \tilde{E}_2 + \tilde{E}_3 + \tilde{E}_4$, where

$$\begin{aligned} \tilde{E}_1 &:= \left\| \begin{bmatrix} 0 \\ B^{-1}(\infty) - B^{-1} \end{bmatrix} \right\|_2^2, \\ \tilde{E}_2 &:= |B^{-1}(\infty)|^2 \|\Lambda_o^{-H} - \Lambda_o(\infty)\|_2^2, \\ \tilde{E}_3 &:= \left\| \begin{bmatrix} W_y N(\Lambda_o^H\Lambda_o)^{-1} \\ 1 - (\Lambda_o^H\Lambda_o)^{-1} \end{bmatrix} B^{-1}(\infty) \right\|_2^2, \\ \tilde{E}_4 &:= \inf_{Q\in\mathbb{R}\mathcal{H}_\infty} \|\Lambda_o R - \Lambda_o(\infty)B^{-1}(\infty) - \Lambda_o Q\tilde{N}\|_2^2, \end{aligned}$$

with $R = B^{-1}Y\tilde{N} + B^{-1}$. Direct calculation yields

$$\tilde{E}_1 = \prod_{i=1}^{N_p} |p_i|^2 - 1,$$

and additionally by application of Lemma 1 we get

$$\tilde{E}_2 + \tilde{E}_3 = (|\Lambda_o(\infty)|^2 - 1) \prod_{i=1}^{N_p} |p_i|^2.$$

Therefore,

$$\tilde{E}_1 + \tilde{E}_2 + \tilde{E}_3 = |\Lambda_o(\infty)|^2 \prod_{i=1}^{N_p} |p_i|^2 - 1 =: E_1.$$

By following similar way in [4] we can show that $\tilde{E}_4 =: E_2$. □

Next, we give two implications of Theorem 2 pertaining to the energy regulation problem.

Corollary 2. *If $W_y = 0$ which implies $\Lambda_o = 1$, then*

$$E_d^* = E_1 + E_2, \tag{15}$$

where

$$\begin{aligned} E_1 &= \prod_{i=1}^{N_p} |p_i|^2 - 1, \\ E_2 &= \sum_{i,j \in \mathbb{N}} \frac{(|s_i|^2-1)(|s_j|^2-1)}{\bar{b}_i b_j (\bar{s}_i s_j - 1)} \times \\ &\quad \left(B^{-1}(\infty) - B^{-1}(s_i)\right)^H \left(B^{-1}(\infty) - B^{-1}(s_j)\right). \end{aligned}$$

Corollary 3. *If $W_y = 0$ and $P(z)$ is strictly proper with relative degree v, minimum phase, and has only one unstable pole $p \in \bar{\mathbb{D}}^c$, then*

$$E_d^* = (p^2 - 1)p^{2v}. \tag{16}$$

Proof: Since P has only one unstable pole, from (15) we get $E_1 = p^2 - 1$. If P has relative degree 1 then $E_2 = (p^2-1)^2$. And if, respectively, P has relative degree 2 and 3 then $E_2 = (p^2-1)^2(1+p^2)$ and $E_2 = (p^2-1)^2(1+p^2+p^4)$. In general, if P has relative degree v then

$$E_2 = (p^2-1)^2 \sum_{k=1}^{v} p^{2(k-1)} = (p^2-1)(p^{2v}-1).$$

Hence, $E_1 + E_2 = (p^2-1)p^{2v}$, which proves (16). □

Theorem 2 shows that the expression of the optimal output regulation performance shares close similarity with that of the optimal energy regulation performance in Theorem 2 of [4], which is reinvented by Corollary 2. Except for the contribution of the outer function Λ_o, the unstable poles and non-minimum phase zeros of the plant give their effects in an analogous fashion. Suppose that $W_y N = (A_N, B_N, C_N, D_N)$ then

$$\Lambda_i \Lambda_o = \begin{bmatrix} W_y N \\ -1 \end{bmatrix} = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right),$$

where $A = A_N$, $B = B_N$, $C = [C_N,\ 0]^T$, and $D = [D_N,\ -1]^T$. Further the transfer function of Λ_o is

$$\Lambda_o(z) = \left(\begin{array}{c|c} A & B \\ \hline D_s F & D_s \end{array} \right),$$

where D_s be an appropriate surjective matrix satisfying $D_s^T D_s = D^T D + B^T \mathcal{P} B$, and $F = (\mathcal{R} + B^T \mathcal{P} B)^{-1}(B^T \mathcal{P} A + \mathcal{S}^T)$, with $\mathcal{P}$ is the solution of discrete-time algebraic Riccati equation

$$\mathcal{P} = A^T \mathcal{P} A + \mathcal{Q} - (A^T \mathcal{P} B + \mathcal{S})(\mathcal{R} + B^T \mathcal{P} B)^{-1}(B^T \mathcal{P} A + \mathcal{S}^T), \tag{17}$$

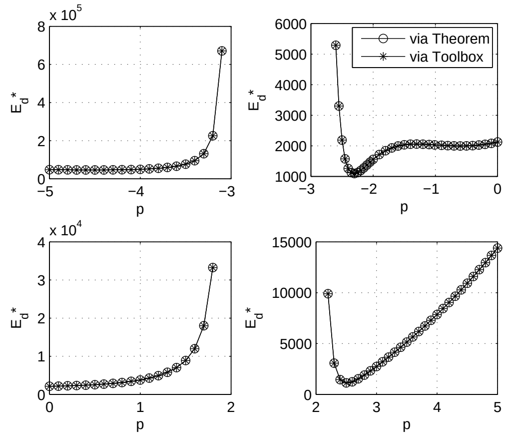

FIGURE 2. E_d^* with respect to p.

where $\mathcal{Q} = C^T C$, $\mathcal{R} = D^T D$, $\mathcal{S} = C^T D$.

We then obtain

$$|\Lambda_o(\infty)|^2 = D_s^T D_s = D_N^T D_N + 1 + B^T \mathcal{P} B.$$

Since $\mathcal{P}$ is positive semi-definite, $|\Lambda_o(\infty)|^2 \geq 1$, which gives a knowledge that $\Lambda_o(\infty)$ makes the regulation performance worse. This fact also confirms that when the plant is stable and minimum phase, its optimal regulation performance, i.e., $E_d^* = |\Lambda_o(\infty)|^2 - 1$ by Theorem 2, is non-negative.

Example 1. *We consider an SISO plant given by*

$$P(z) = \frac{(z-2)(z+3)}{(z-5)(z-p)},$$

which has non-minimum phase zeros at $z = 2$ and $z = -3$, and unstable poles at $z = 5$ and possibly at $z = p$. Fig. 2 plots Theorem 2 based computation (circled-line) and toolbox-based computation (stared-line) for p from -5 to 5. Here we set $W_y(z) = 1$.

5. Time Delay Systems

We consider the following continuous-time delay system

$$P(s) = \frac{P_0(s)}{s-\lambda} e^{-\tau s}, \tag{18}$$

where $P_0(s)$ is biproper, minimum phase, stable, and possibly single-input multiple-output, $\lambda > 0$ is the only unstable pole of $P(s)$, and $\tau \geq 0$ indicates the delay time. We minimize the performance index

$$E_c := \int_0^\infty |u(t)|^2 \, dt, \tag{19}$$

with respect to an impulse disturbance signal.

Proposition 1. *Let the plant $P(s)$ is given by (18). Then*

$$E_c^* = 2\lambda e^{2\lambda\tau}.$$

Proof: We follow an indirect way to prove, i.e., by using continuity properties. It is known that the zero-order hold operation with sampling time T will convert the continuous-time delay plant $P(s)$ onto its delta-type counterpart $P(\delta)$ as follows

$$P(\delta) = \frac{P_0(\delta)}{\delta - \rho}\delta^{-\frac{\tau}{T}} = \frac{P_0(\delta)}{\delta^{\tau/T}(\delta - \rho)},$$

in which $P(\delta)$ is a strictly proper plant with relative degree $v = \tau/T+1$. Then from Corollary 3 and Theorem 3 of [4] we obtain its optimal energy regulation performance in delta domain as follows

$$E_\delta^* = \frac{((T\rho+1)^2-1)(T\rho+1)^{2(\frac{\tau}{T}+1)}}{T}.$$

The corresponding continuous-time optimal performance then can be recovered by taking the sampling time T tend to zero, i.e., $E_c^* = \lim_{T\to 0} E_\delta^* = 2\lambda e^{2\lambda\tau}$. It holds since $\rho = (e^{\lambda T}-1)/T$. □

Alternatively, by using the first order Padé approximation we may approximate the delay part as follows

$$e^{-\tau s} \approx \frac{2/\tau - s}{2/\tau + s}.$$

Hence,

$$P(s) \approx P_p(s) = \frac{P_0(s)}{s-\lambda}\frac{2/\tau - s}{2/\tau + s}, \tag{20}$$

which has one unstable pole at λ and one non-minimum phase zero at $2/\tau$. From Corollary 1 we get the optimal regulation performance of plant $P_p(s)$,

$$E_p^* = 2\lambda + \frac{16\lambda^2}{\tau(2/\tau - \lambda)^2}. \tag{21}$$

We can confirm that the Padé approximation works well only for the smaller value of λ.

6. Conclusion

In this paper, we have examined the $\mathcal{H}_2$ output regulation problem for SIMO LTI feedback control systems. We derive the closed-form solutions of the optimal regulation performance for unstable/non-minimum phase continuous-time and discrete-time systems. The direct implication of our main results covers the energy regulation problem and that of minimum phase time delay systems.

In general, our results confirm that the minimal output regulation performance depends upon plant unstable poles, plant non-minimum phase zeros, certain outer factors, and plant gain for continuous-time systems.

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