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RANDOM PARAMETER MODELS OF FERTILIZER RESPONSE FOR CORN USING SKEWED DISTRIBUTIONS

Mohammad Masjkur

Department of Statistics, Bogor Agricultural University, Indonesia

ABSTRACT

Random parameter models have been found to better determine the optimum dose of fertilizer than fixed parameter. However, a major restriction of it is the normality assumption.. The purpose of this study the introduction of random parameter models of fertilizer response using skewed distributions from a Bayesian perspective. The method is applied to data sets of multilocation trials of potassium fertilization on corn. We compare the Linear Plateau, Spillman-Mitscherlich, and Quadratic random parameter models with different random errors distribution assumption, i.e. as normal, skew-normal, Student-t and Skew-t distribution using the Deviance Information Criterion (DIC). The results show that the smallest DIC value is obtained for the normal linear plateau model compare with the other models. The correlation between observed and fitted values was significant.

Key words **:** *fertilizer response model, mixed effects, skewed distributions, DIC.*

INTRODUCTION

Background

Many linear and nonlinear functions have been used for describing multi-environment crop response to fertilizer, such as linear plateau and quadratic functions. The model parameters usually estimated using least squares method assuming that the model has a fixed effect and the random error terms were independent and normally distributed and have a constant variances. However, some authors suggested the approach was unrealistic ignoring the variability and correlation that probably exist between site-years (Wallach, 1995; Makowski and Wallach, 2002; Makowski and Lavielle, 2006).

The alternative model was estimating parameters of fertilizer response model using mixed effects approach. This approach allowing parameters to have a random effect that represent between site-years variability, heterogenous variance and correlation that probably exist between observations. Some studies showed that a random parameter model approach was statistically better than a fixed parameter model version for determining optimum doses of fertilizer recommendation (Makowski et al., 2001; Makowski and Wallach, 2002; Tumusiime et al., 2011; Boyer et al., 2013).

The assumptions of this model type are the response functions are the same for all site-years, but the value of parameters vary between site-years. The model parameters and random errors usually considered as a normal random variable (Makowski and Wallach, 2002; Makowski and Lavielle, 2006; Tumusiime et al., 2011; Boyer et al., 2013). However, normality (symmetric) assumptions of this model type may be too restrictive as it suffers from the lack of robustness against departures from the normal distribution, and thus may not provide an accurate estimation.

Day (1965) found that the field crop yield distributions are in general nonnormal and nonlognormal. The degree of skewness and kurtosis depends upon the specific crops and on the amount of available nutrients. The weather random effects also could result in positive or negative skewed probability functions.

Previous research has proposed using a flexible classes of random effects density of mixed model such as Student's-*t* (Pinheiro et al., 2001), skew-normal (Arellano-Valley et al., 2005; Huang et al., 2011), skew-*t* (Jara et al., 2008; Dagne, 2013), beta distribution (Ouedraogo and Brorsen, 2014), Normal

Independent (NI) distributions (Meza et al., 2012; Lachos, et al., 2013; De la Cruz, 2014), and Skew-Normal Independent (SNI) distributions (Lachos et al., 2009; 2010). The flexibility of these distributions was allowing to fit observations with a high presence of skewness and/or heavy tails and more efficient than the normal distribution approach.

Estimation of random parameter fertilizer response model usually based on maximum likelihood approach. However, such an approach can lead to unbiased parameter estimates for models that are linear in the parameters. But, for models that are nonlinear in the parameters, such estimation methods have the possibility of nonunique optimal point. Convergency of maximum likelihood estimation can be difficult to obtain even with careful scaling and good starting values (Tumusiime et al., 2011; Brorsen, 2013).

Brorsen (2013) advocated Bayesian estimation of random parameter fertilizer response model for determining optimum doses of fertilizer. Bayesian estimation methods offer two major advantages over frequentist approaches such as maximum likelihood. First, the results are valid in small samples, which is the case of crop yield response to fertilizer. Second, convergence of nonlinear estimation methods is not an issue with Bayesian methods.

The purpose of this study was to estimate random parameter model of fertilizer response using skew-elliptical distributions from a Bayesian perspective.

LITERATURE REVIEW

A. Bayesian random parameter models with skewed distribution

To account for the skewness and heavy tailed observed in the data, the random error in the random parameter model can be assumed to follow a skew-elliptical (SE) distribution (Sahu et al., 2003; Huang and Dagne, 2012, Chen, 2012, Dagne, 2013). In the SE family, skew-normal (SN), normal and Student-t distribution are all a special case of skew-t (ST) distribution. A general random parameter model of fertilizer response with an ST distribution under the Bayesian approach can be expressed as:

$$\boldsymbol{y}_i = \boldsymbol{g}_i\,(\boldsymbol{\beta}_i\,, \boldsymbol{d}_i) + \varepsilon_i, \varepsilon_i \sim ST_{n_i,v}(0, \boldsymbol{\Sigma}, \boldsymbol{\Delta}) \quad [1]$$
$$\boldsymbol{\beta_i} = \boldsymbol{h}\,(\boldsymbol{\beta}, \boldsymbol{b_i}\,), \boldsymbol{b_i} \sim N(0, \boldsymbol{\Sigma}_b)$$

y_i = (y_{i1}, ……., y_{in_i})^T with y_{ij} being the response value for the *j*th measurement of the yield response in the *i*th site-year when the fertilizer dose d_{ij} is applied (i=1, 2, ….,n, j=1,2, …, n_i). $\boldsymbol{\beta_{ij}}$ are site-year-specific parameter vector and $\boldsymbol{\beta}$ is population parameter vector, *g(.)* and *h(.)* are linear or nonlinear known parametric functions, $\boldsymbol{b_i}$ is normal random effect vector with $\boldsymbol{\Sigma}_b$ being an unstructured covariance matrix. The vector of random errors ε_i = (ε_{i1}, ……., ε_{in_i})^T follows a multivariate ST distribution with degrees of freedom *v*, within subject covariance matrix $\boldsymbol{\Sigma}$, and assumed $\boldsymbol{\Sigma} = \sigma^2 \boldsymbol{I_{n_i}}$, and unknown n_i x n_i skewness diagonal matrix such that $\boldsymbol{\Delta}$ = diag (δ_{i1}, ……., δ_{in}), skewness parameter vector δ_i = (δ_{i1}, ……., δ_{in})^T. In particular, if δ_{i1}=…….=$\delta_{in} \triangleq \delta$, then Δ= $\delta\boldsymbol{I}_{n_i}$ and $\delta_i = \delta 1_{n_i}$, where 1_{n_i} = (1, …….,1)^T, indicating skewness of overall data set.

To implement an MCMC procedure to above model, by introducing one n_i x 1 random vector ω_i, based on the stochastic representation, the model can be hierarchically formulated as follows,

$$\boldsymbol{y}_i|\boldsymbol{b}_i\,,$$
$$\omega_i \sim t_{n+v}\left(g(\boldsymbol{\beta}_i\,, \boldsymbol{d}_i) + \delta\omega_i\,, \omega_i\,\sigma^2 \boldsymbol{I}_{\boldsymbol{n}_i}\right), \quad [2]$$
$$\omega_i \sim t_{n+v}\left(0, \boldsymbol{I}_{\boldsymbol{n_i}}\right),\ \boldsymbol{I}\,(\omega_i > 0),$$
$$\boldsymbol{b_i} \sim N(0, \boldsymbol{\Sigma}_b)$$

where $\omega_i = (v + \omega_i^T \omega_i)/(v + n_i)$, $t_{n,v}$ (μ, *A*) denotes the n_i variate Student-t distribution with parameters μ, *A*, and degrees of freedom v, $\boldsymbol{I}$ $(\omega_i > 0)$ is an indicator function and ω_i =|X_0| with $X_0 \sim t_{n+v}\left(0, \boldsymbol{I_{n_i}}\right)$.

The unknown population parameters in the model are $\theta = \{\beta, \sigma^2, \Sigma_b, \delta, v\}$, and assumed they are independent of one another. Under Bayesian framework, the prior distributions for unknown parameters are as follows,

$$\beta \sim N(\beta_0, \boldsymbol{\Lambda}), \qquad \sigma^2 \sim IG(\omega_1, \omega_2),$$
$$\boldsymbol{\Sigma_b} \sim IW(\boldsymbol{\Omega}, v), \qquad [3]$$
$$\delta \sim N(\boldsymbol{0}, \gamma),$$
$$v \sim Exp(v_0) I(v > 2)$$

where the mutually independent Normal (N), Inverse Gamma (IG), Exponential (Exp), and Inverse Wishart (IW) prior distribution are chosen to facilitate computations. The super-parameter matrices $\boldsymbol{\Lambda}$ and $\boldsymbol{\Omega}$ can be assumed to be diagonal for convenient implementation.

Let π (.) be a prior density function, so $\pi(\boldsymbol{\theta}) = \pi(\beta)\pi(\sigma^2)\pi(\boldsymbol{\Sigma_b})\pi(v)\pi(\delta)$. Denote the observed data by $D = \{y_i, i = 1, \dots, n\}$, and $f(.|.)$ as a conditional density function. Based on Bayesian inference, the posterior density of $\boldsymbol{\theta}$ is proportional to the observed data and prior distribution as:

$\int(\boldsymbol{\theta}|D)\,\{\prod_i^n \int\int(y_i|b_i,\ \omega_i;\beta,\sigma^2,v,\delta)\int(\omega_i|\omega_i > 0)\int(b_i|\Sigma_b)db_i\}\pi(\theta)$ [4]

In general, the integral in [4] is of high dimension and does not have any closed form. Analytic approximations to the integral may not be sufficiently accurate. Therefore, it is prohibitive to directly calculate the posterior distribution of $\boldsymbol{\theta}$ based on the observed data. As an alternative, MCMC procedures can be used to sample based on [4] by the Gibbs sampling.

METHODOLOGY

Data

The study using data sets of multilocation trials of potassium fertilizer on corn (Syafruddin et al., 2004; Sutriadi et al., 2008). Each trial consists of five levels of potassium fertilizer treatment. The response measured was corn grain dry weight (t/ha). The corn grain yield responses obtained with different potassium fertilizer treatments was shown in Figure 1.

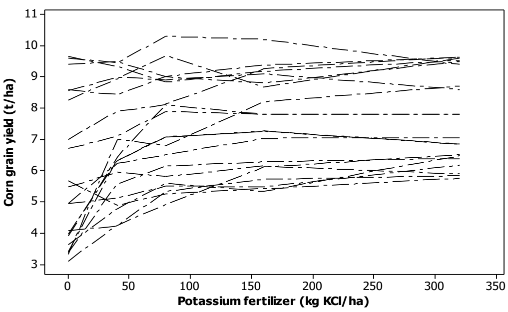

Figure 1. The corn yield response to applied potassium

Response functions

In this paper, three response functions are considered including a linear-plus plateau (LP), Spillman-Mitscherlich (SM) and quadratic functions (Q). Under the general layout as model [1], the random parameter models of fertilizer response can be expressed as follow,

1. The stochastic linear plateau response model

$Y_{ij} = \min\big(\alpha_{1i} + (\alpha_{2i}d_{ij};\ \alpha_{3i}\ \big) + \varepsilon_{ij}$ [5]

$\varepsilon_i \sim ST_{n_i,v}\big(0, \boldsymbol{\sigma^2 I_{n_i}},\ \delta\boldsymbol{I_{n_i}}\big)$

$\alpha_{1i} = \alpha_1 + b_{1i}, \alpha_{2i} = \alpha_2 + b_{2i}, \alpha_{3i} = \alpha_3 + b_{3i}, \alpha = (\alpha_1, \alpha_2, \alpha_3)^T$ and $\boldsymbol{b_i} = (b_{1i}, b_{2i}, b_{3i})^T \sim N_3(0, \boldsymbol{\Sigma}_b)$

2. The stochastic Spillman-Mitscherlich response model

$Y_{ij} = \beta_{1i} - \beta_{2i}\exp(-\beta_{3i}\, d_{ij}) + \varepsilon_{ij}$ [6]

$\varepsilon_i \sim ST_{n_i,v}\big(0, \boldsymbol{\sigma^2 I_{n_i}},\ \delta\boldsymbol{I_{n_i}}\big)$

$\beta_{1i} = \beta_1 + b_{1i}, \beta_{2i} = \beta_2 + b_{2i}\ ,\ \beta_{3i} = \beta_3 + b_{3i}, \beta = (\beta_1,\ \beta_2,\ \beta_3)^T$,and $\boldsymbol{b_i}$ $(b_{1i},\ b_{2i},\ b_{3i})^T \sim N_3(0, \boldsymbol{\Sigma}_b)$

3. The stochastic quadratic response model

$Y_{ij} = \gamma_1 + \gamma_2\, d_{ij} - \gamma_3 d_{ij}^2 + \varepsilon_{ij}$ [7]

$\varepsilon_i \sim ST_{n_i,v}\big(0, \boldsymbol{\sigma^2 I_{n_i}},\ \delta\boldsymbol{I_{n_i}}\big)$

$\gamma_{1i} = \gamma_1 + b_{1i}\ , \gamma_{2i} = \gamma_2 + b_{2i}\ ,\ \gamma_{3i} = \gamma_3 + b_{3i}, \gamma = (\gamma_1,\ \gamma_2,\ \gamma_3)^T$,and $\boldsymbol{b_i}$ $(b_{1i},\ b_{2i},\ b_{3i})^T \sim N_3(0, \boldsymbol{\Sigma}_b)$

Statistical Analysis

The data sets was used to explore the best fit among the random parameter model of fertilizer response and different random errors distribution assumption such as normal, skew-normal, Student-t and Skew-t distribution. The model fit was selected using Deviance Information Criterion (DIC). Plots and correlation of observed values vs. fitted values, were also examined to explore goodness-of-fit in the model comparisons.

The following independent priors were considered to perform the Gibbs sampler,

$\boldsymbol{\alpha_k} \sim N(0, 100), \boldsymbol{\beta_k} \sim N(0, 100)$,
$\boldsymbol{\gamma_k} \sim N(0, 100)$,k=1,2,3,
$\sigma^2 \sim IG(0.01, 0.01),\ \Sigma_b \sim IW(\Omega, 5),$ and Ω is diagonal matrix with diagonal elements being 0.01, $\delta \sim N(0, 0.01)$, and $v \sim Exp(0.5)I(2,)$. Considering these prior densities we generated two parallel independent runs of the Gibbs sampler chain with size 20 000 for each parameter with the first 10 000 times as burn in runs. The MCMC sampler was implemented using OpenBUGS software.

RESULT AND DISCUSSION

Figure 2 shows histogram and normal Q-Q plot of corn yield data for 20 experiments. There are apparent non-normal (heavy tailed) pattern of corn yield data. The skewness value is slightly negative (-0.17) and standard error was 0.24. The ratio between skewness and standard error was 0.71.

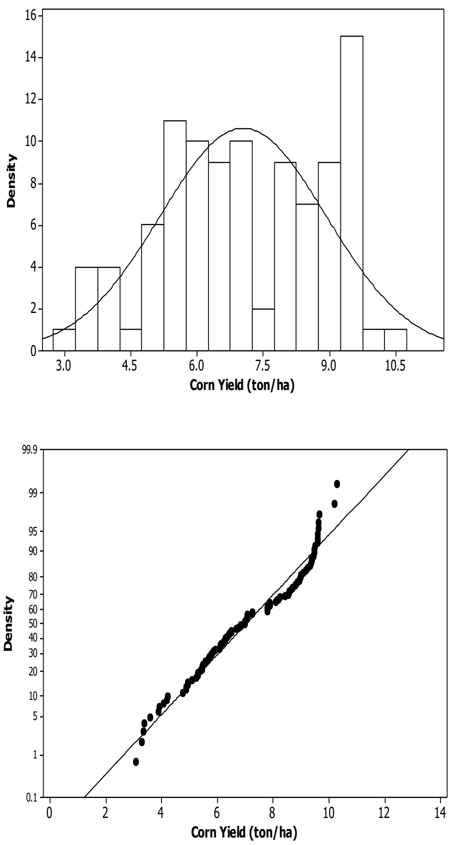

Figure 2. Histogram and Normal Q-Q plot of corn yield data

The population posterior mean (PM), the corresponding standard deviation (SD) and 95% credible interval (CI) for fixed-effect parameters of the four distributions of linear plateau model are presented in Table 1.

The estimates were substantially different between four distribution assumptions. All of the parameter estimates were significant since the 95% CIs don't include zero. The skewness parameter δ was not significantly negative in SN and ST, since the 95% CIs include zero. Compared to the model with SN or ST distribution assumption, the models with an normal or Student-*t* fit the data better. The DIC values were 123.0 (normal) vs. 415.4 (SN), and 153.5 (Student-*t*) vs. 402.2 (ST) indicating that consideration of a skewness does not improve the model fit. Also, considering heavy tailed distribution does not improve the model fit since the DIC values of the normal model was smaller than Student-t model. Therefore, the normal linear plateau model was the best model for corn yield data.

Tabel 1. Parameter estimate of Linear Plateau model for corn yield data

LP		α_1	α_2	μ_p	σ^2	δ	DIC
Normal	PM	5.82	2.92	7.36	0.35	-	123.0
	L_{CI}						
	U_{CI}	4.13	0.15	6.57	0.14		
	SD						
		6.93	21.16	8.03	0.49		
		0.90	5.32	0.39	7.46		
Skew-	PM	6.23	5.01	7.53	0.24	-	415.4
Normal	L_{CI}				0.05	0.04	
	U_{CI}	4.79	0.04	6.87	0.38	-	
	SD		25.33		6.38	0.25	
		7.79	6.78	8.33		0.16	
		0.72		0.35		0.12	
Student-	PM	5.87	3.54	7.37	0.19	-	153.5
t	L_{CI}		0.05				
	U_{CI}	4.79	19.33	6.60	0.09		
	SD		5.50				
		6.97		7.95	0.38		
		0.59		0.33	0.07		
Skew-t	PM	6.17	3.26	7.24	0.24	-	402.2
	L_{CI}	4.88				0.05	
	U_{CI}	7.02	1.14	6.53	0.14	-	
	SD	0.53	16.13	7.67	0.40	0.24	
			4.28	0.32	0.07	0.14	
						0.09	

Table 2 present the population posterior mean (PM), the corresponding standard deviation (SD) and 95% credible interval (CI) for fixed-effect parameters of the four distributions of Spillman-Mitscherlich model.

Tabel 2. Parameter estimate of Spillman-Mitscherlich model for corn yield data

SM		β_1	β_2	β_3	σ^2	δ	DIC
Normal	PM	7.094	1.722	4.946	0.7059	-	176.7
	L_{CI}	0.5215					
	U_{CI}		0.9335	0.5485	0.1472		
	SD	6.243					
			2.484	18.98	0.6109		
		7.782	0.7552	5.727	4.028		
Skew-	PM	7.259	1.11	4.385	0.2874	0.05094	466.7
Normal	L_{CI}					-0.2621	
	U_{CI}	6.629	0.4311	0.5048	0.06886	0.2624	
	SD				0.4033	0.2126	
		7.952	1.932	18.66	1.386		
		0.4289	0.5972	5.325			
Student-t	PM	7.3	1.423	7.303	0.29	-	226.0
	L_{CI}						
	U_{CI}	6.695	0.8408	1.367	0.1602		
	SD						
		7.888	1.938	21.16	0.4775		
		0.3073	0.276	5.346	0.08151		
Skew-t	PM	7.326	1.483	8.356	0.3518	-0.04126	494.4
	L_{CI}					-0.2708	
	U_{CI}	6.632	0.8747	1.419	0.2072	0.1887	
	SD					0.1166	
		7.956	2.111	22.51	0.551		
		0.334	0.3106	5.857	0.08697		

All of the parameter estimates were significant since the 95% CIs don't include zero. The estimates were substantially different between four distribution assumptions. The skewness parameter δ was not significant in SN and ST, since the 95% CIs include zero. Compared to the model with SN or ST distribution assumption, the models with an normal or Student-*t* fit the data better. The DIC values were 176.7 (normal) vs. 466.7 (SN), and 226.0 (Student-*t*) vs. 494.4 (ST). However, the DIC values of a normal model was smaller than the Student-*t* model. It indicates that consideration of a skewness and heavy tailed distribution does not improve the model fit. Therefore, the normal Spillman-Mitscherlich model was the best model for corn yield data.

Table 3 present the population posterior mean (PM), the corresponding standard deviation (SD) and 95% credible interval (CI) for fixed-effect parameters the four distributions of quadratic model.

Table 3. Parameter estimate of quadratic model for corn yield data

Q		γ_1	γ_2	γ_3	σ^2	δ	DIC
Normal	PM	4.296	3.506	4.37	0.288		217.2
	L_{CI}	-11.72	- 0.03751	1.8	0.1824		
	U_{CI}	17.25	7.043	6.94	0.4515		
	SD	8.613	3.514	2.57	0.07869		
Skew-Normal	PM	7.259	1.11	4.385	0.2874	0.05094	466.7
	L_{CI}	6.629	0.4311	0.5048	0.06886	-0.2621	
	U_{CI}	7.952	1.932	18.66	0.4033	0.2624	
	SD	0.4289	0.5972	5.325	1.386	0.2126	
Student-t	PM	6.444	3.187	2.733	0.2001	-	221.9
	L_{CI}	2.698	2.048	1.711	0.09413		
	U_{CI}	11.04	4.32	3.754	0.3579		
	SD	2.278	1.093	1.021	0.06852		
Skew-t	PM	2.018	6.427	2.597	0.247	-0.1126	488.2
	L_{CI}	-3.351	3.556	1.676	0.1367	-0.3649	
	U_{CI}	8.177	9.286	3.519	0.4094	0.1386	
	SD	3.226	2.84	0.9214	0.06903	0.1275	

The estimates were substantially different between four distribution assumptions. Except for Normal and ST-model, the parameter estimates of intercept were significant since the 95% CIs don't include zero. The parameter estimates of linear coefficient were significant, except for Normal-model since the 95% CIs include zero. The parameter estimates of quadratic coefficient were significant for all models. The skewness parameter δ was not significant in SN and ST, since the 95% CIs include zero. Compared to the model with SN or ST distribution assumption, the models with an normal or Student-*t* fit the data better. The DIC values were 217.2 (normal) vs. 221.9 (Student-t), while 466.7 (SN) vs. 488.2 (ST). The DIC values sequentially were Normal < Student-t < Skew-normal < Skew-t. The normal random model was smallest than the other models. Therefore, the normal quadratic model was the best model for corn yield data.

Table 4 present the comparison among the linear-plus plateau (LP), Spillman-Mitscherlich (SM) and quadratic functions (Q) with four distributions. The normal random parameter linear plateau model has DIC value smallest compare with the other models. The correlation between observed and fitted values of corn yield data was significant (r=0.983; P=0.000) and fit well to the observed data (Figure 3).. Among all models the normal linear plateau model was the best model for corn yield data.

Table 4. DIC values of the LP, SM and Q models with four distributions.

Distribution	DIC		
	LP	SM	Q
Normal	123.0	176.7	217.2
SN	415.4	487.7	466.7
Student-t	153.5	226	221.9
ST	402.2	494.4	488.2

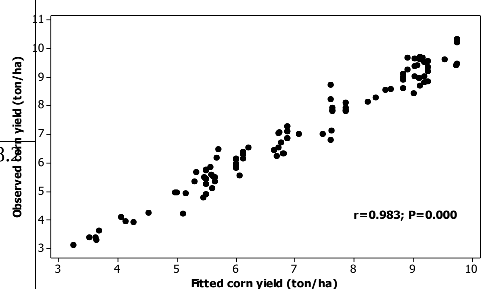

Figure 3. Scatter diagram of observed and fitted corn yield data for Normal-LP model

In this paper we compare the four distributions of random error of random parameter model to account for asymmetric and/or heavy tailed distribution that could be observed in the data. The four distributions include normal, Student-t, Skew-normal and Skew-t distribution. The random parameter models consist of linear plateau, Spillman-

Mitscherlich and quadratic model. However, results show that consideration of skewness and/or heavy tailed distribution does not improve the model fit for corn yield data.

Apparently the corn yield data does not exhibit high degree of skewness and/or heavy tails. The ratio between skewness and standard error was smaller than 2. If the ratio between skewness and standard error was greater than 2, the data may be regarded as having unignorable skewness (Chen, 2012).

Another possibility that the nonnormal (heavy tailed) observed in the data may be caused by random effects density, not by random error term. According to Bandyopadhyay et al. (2012) that one (or both) of the (within-subject) random error and (between-subject) random effects might contribute to the ‘shift from normality’. Then, further research may be consider the skew-elliptical distribution of random effects density.

CONCLUSION

In this case the best model for corn yield data was stochastic linear plateau with normal random error distribution.

REFERENCES

Arellano-Valle, R.B., Bolfarine, H., Lachos, V.H., 2005. Skew-normal linear mixed models. Journal of Data Science 3, 415–438.

Bandyopadhyay, D., Lachos, V., Castro, L. M., and Dey, D. (2012), Skewnormal/independent linear mixed models for censored responses with applications to HIV viral loads,” *Biometrical Journal*, 54, 405–425.

Boyer, C.N., Larson, J.A., Roberts, R.K., McClure, A.T., Tyler, D.D., and Zhou, V. (2013). Stochastic corn yield response functions to nitrogen for corn after corn, corn after cotton, and corn after soybeans. *Journal of Agricultural and Applied Economics*, 45,4:669-681.

Brorsen, B.W. (2013). “Using Bayesian Estimation and Decision Theory to Determine the Optimal Level of Nitrogen in Cotton.” Selected Paper, Southern Agricultural Economics Association, Orlando, Florida.

Chen, R. (2012). Bayesian Inference on Mixed-effects Models with Skewed Distributions for HIV longitudinal Data. Graduate Theses and Dissertations. University of South Florida.

Day, H.R. 1965. “Probability Distributions of Field Crop Yields.” *Journal of Farm Economics* 47(3):713-741.

Dagne, Getachew A. 2013. Bayesian Inference for Skew-Normal Mixture Models With Left-Censoring. *J Biopharm Stat*. ; 23(5): 1023–1041.

De la Cruz, Rolando. (2014). Bayesian analysis for nonlinear mixed-effects models under heavy-tailed distributions. *Pharmaceut. Statist.* 13: 81–93.

Huang, Yangxin; Chen, Ren; and Dagne, Getachew. 2011. Simultaneous Bayesian Inference for Linear, Nonlinear and Semiparametric Mixed-Effects Models with Skew-Normality and Measurement Errors in Covariates. *The International Journal of Biostatistics*: Vol. 7 : Iss. 1, Article 8.

Huang, Y. and G A. Dagne. 2012. Simultaneous Bayesian inference for skew-normal semiparametric nonlinear mixed-effects models with covariate measurement errors. Bayesian Anal. 7(1): 189–210.

Jara, A., Quintana, F., San Martin, E., 2008. Linear mixed models with skew-elliptical distributions: a Bayesian approach. Computational Statistics and Data Analysis 52, 5033–5045.

Lachos, V. H., Dipak K.Dey, Vicente G.Cancho. (2009) Robust linear mixed models with skew-normal independent distributions from a Bayesian perspective. *Journal of Statistical Planning and Inference* 139:4098-4110.

Lachos, V. H., Pulak Ghosh and Reinaldo B. Arellano-Valle. (2010). Likelihood Based Inference For Skew-Normal Independent Linear Mixed Models. *Statistica Sinica* 20: 303-322.

Lachos, V. H., Luis M. Castro, Dipak K. Dey. (2013). Bayesian inference in nonlinear mixed-effects models using normal independent distributions. *Computational Statistics and Data Analysis* 64: 237–252.

Makowski, D. and Wallach, D. (2002). It pays to base parameter estimation on a realistic description of model errors. *Agronomie* 22:179–89.

Makowski, D. and Lavielle M. (2006). Using SAEM to estimate parameters of response to applied fertilizer. *Journal of*

Agricultural , Biological, and Environmental Statistics 11:45-60.

Meza C, Osorio F, De la Cruz R. (2012). Estimation in nonlinear mixed-effects models using heavy-tailed distributions. *Statistics and Computing.*; 22:121–139.

Pinheiro, J. C., Liu, C. H. and Wu, Y. N. (2001). Efficient algorithms for robust estimation in linear mixed-effects models using a multivariate t-distribution. *J. Comput. Graph. Statist.* **10**, 249-276.

Sahu SK, Dey DK, Branco MD. 2003. A new class of multivariate skew distributions with applications to Bayesian regression models. The Canadian Journal of Statistics.; 31:129–150.

Syafruddin, Saidah, A. Ardjanhar, C. Manopo, and D. Setyiorini. (2004). Study of P and K fertilizer recommendation for corn in upland soils. Agricultural Technology Development Institute. Central Sulawesi.

Sutriadi M. T., D. Setyorini, D. Nursyamsi, and A. M. Murni. (2008). Determining potassium fertilizer requirement for corn using soil test potassium on *Typic Kandiudox*. J. Trop. Soils, 13 (3) : 179-187.

Tumusiime, E., Brorsen, B.W., Mosali, J., Johnson, J., Locke, J. and Biermacher, J.T. (2011). Determining optimal levels of nitrogen fertilizer using random parameter models. *Journal of Agricultural and Applied Economics* 43(4) :541–552.

Wallach, D. (1995). Regional optimization of fertilization using a hierarchical linear model. *Biometrics* 51 : 338–346.