A FAST COMPUTATION FOR EIGENVALUES OF CIRCULANT MATRICES WITH ARITHMETIC SEQUENCE

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Abstract

In this article, we derive simple formulations of the eigenvalues, determinants, and also the inverse of circulant matrices whose entries in the first row form an arithmetic sequence. The formulation of the determinant and inverse is based on elementary row and column operations transforming the matrix to an equivalent diagonal matrix so that the formulation is obtained easily. Meanwhile, for the eigenvalues formulation, we simplify the known result of formulation for the general circulant matrices by exploiting the properties of the cyclic group induced by the set of all roots of $x^n - 1 = 0$ as the set of points in the unit circle in the complex plane, and also by considering the specific property of arithmetic sequence. Then, we construct an algorithm for the eigenvalues formulation. This algorithm shows a better computation compared to the previously known result for the general case of circulant matrices.

Keywords: Circulant matrix, Arithmetic Progression, Eigenvalues, Determinant, Inverse, Cyclic group.

1 Introduction

Many applications of circulant matrices come from mathematics problems such as numerical analysis, linear differential equations, cryptography, operator theory, and many others. Therefore, these could also be associated with computer science and engineering. Those are because of the good structure of the circulant matrix which the computation methods of the eigenvalues, eigenvectors, determinants, and also the inverse can be formulated explicitly.

Recently, with various specializations, the above problems have been discussed in many papers. Here, we refer to some papers on those. The inverse dan determinant formulations of circulant matrices involving a geometric sequence were studied by Bueno [6]. Shen et al. [20] proposed some conditions for the circulant matrix invertibility with an entry of the Fibonacci and Lucas sequence, the determinant and inverse formulations for those matrices are also derived. The generalization of those works by changing the matrix entry in the first row of the *k*-Fibonacci and *k*-Lucas sequence was done by Jiang et al. [12]. Then, Jiang and Li [11] continued the results of Jiang by changing the circulant

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matrices to the G-circulant and the left circulant matrices. Li et al. see in [14], in the same year, proposed an explicit formulation of the determinants of circulant and also left circulant matrices with Tribonacci and generalized Lucas numbers.

Further investigation on the explicit formulation of the determinant and inverse of circulant matrices continued by indroducing the entry of Tribonacci and changing the matrix structure to skew circulant, presented by Jiang and Hong in [9]. Next, a computational method by applying a symbolic algorithm to compute the determinant and inverse of bordered tridiagonal matrices was proposed by Jia and Li in [10]. Radicic [19] followed the study of k-circulant matrices with geometric sequence, while Bozkurt and Tam [5] were interested in r-circulant matrices associated with a number sequence. Most recently, similar problems can be seen in [4], [17], [3], [13], [21], and [16].

Now in this paper, we propose the formulations of the eigenvalues, inverse, and also the determinant for the circulant matrices involving the arithmetic sequence in a simpler way than in the formulations for the general case. The formulation method for determining the inverse and determinant is based on a series of elementary row and column operations which are directed to an equivalent diagonal matrix, and the result can be stated in one theorem. Meanwhile, for the eigenvalues formulation, we simplify the known result formulation of general circulant matrices by exploiting the properties of the cyclic group induced by the set of all roots of $x^n - 1 = 0$ as the set of points in the unit circle in the complex plane, and also by considering the speciality of the arithmetic sequence These topics are specific cases of the results by Radicic [18]. Here, however, we have a different approach in the methods especially in the proofs, and also our results are more specific and easier for the computation aspect. The outline of this paper is presented as follows.

We present a short review for the notion of the general circulant matrix in Section 2, including previous results related to its determinant, inverse, and eigenvalues. At the end of this section, we also discuss about the notion of arithmetic sequence especially connected with the definition of its circulant matrix. Section 3 contains a theorem and its proof which describes the simple formulation for the inverse and determinant of the matrix defined in Section 2. The eigenvalues formulation is the main result of this paper, and we construct an algorithm of this formulation which shows a better computation compared to the previously known result of the case general, presented in Section 4 and 5. In Section 6, we close the paper by giving a conclusion.

2 Circulant Matrices with Arithmetic Sequence

We review the topic of general circulant matrix in the first subsection, and also present some known results associated with the formulation of the determinants, inverse, and eigenvalues. Then, we review the notion of arithmetic progression in the last subsection and also discuss some of its properties which are connected to the subsequent sections.

2.1 Known Results for General Circulant Matrices

For any sequence of numbers, $c_0, c_1, \cdots, c_{n-2}, c_{n-1}$, the $n \times n$ circulant matrix with the sequence in the first row, notation $\mathrm{Circ}(c_0, c_1, \cdots, c_{n-2}, c_{n-1})$, is defined as

$$\mathrm{Circ}(c_0, c_1, \cdots, c_{n-2}, c_{n-1}) = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & c_1 \\ c_1 & c_2 & \cdots & c_{n-1} & c_0 \end{pmatrix}.$$

Let C be the $\mathrm{Circ}(c_0, c_1, \cdots, c_{n-2}, c_{n-1})$, λ_k be the eigenvalues, and for $k = 0,1,2,\ldots, n-1$, let v_k be the corresponding eigenvectors of λ_k. The well-known formulation of λ_k and v_k (see for examples in ([7], [2], [1]) are

$$\lambda_k = \sum_{j=0}^{n-1} c_j \omega^{jk} \quad \text{and} \quad v_k = \left(1, \omega^k, \omega^{2k}, \cdots, \omega^{(n-2)k}, \omega^{(n-1)k}\right) \tag{1}$$

where $\omega = e^{\frac{2\pi}{n}} = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$ with $i = \sqrt{-1}$. In this case, the set $S = \{1, \omega, \omega^2, \cdots, \omega^{n-1}\}$ is a subgroup of the multiplication group of complex numbers $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. In fact, S is cyclic and ω is a generator of S, and we can see that all the elements of S are solutions of $x^n - 1 = 0$. For the sake of simplification, we write down Equation (1) as a matrix multiplication

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{(n-1)2} & \cdots & \omega^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{pmatrix} = \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n-1} \end{pmatrix} \tag{2}$$

It is clear from Equation (1), we obtain that the determinant and inverse formulation of C are

$$\det(C) = \prod_{k=0}^{n-1} \sum_{j=0}^{n-1} c_j \omega^{jk} \quad \text{and} \quad C^{-1} = \mathrm{Circ}(u_0, u_1, \cdots, u_{n-2}, u_{n-1}) \tag{3}$$

where $u_i = \frac{1}{n}\sum_{k=0}^{n-1} \lambda_k \omega^{-ij}$ for $i = 0,1, \cdots n-1$. Increasing the value of n implies the computation of those formulas is not efficient to be implemented. This is because of applying complex number arithmetic even if the elements of the matrix are real numbers. But, if the formation of $c_0, c_1, \cdots, c_{n-2}, c_{n-1}$ has a good structure, for instance, the sequence formulated by recurrence relation, then we have a big chance to simplify to get better formulations for the determinant, inverse, and eigenvalues of C. These become interesting research topics over the last decades which mostly focus on the determinant and inverse. Based on those background topic research, in this paper; we are focusing on observing simplification of the eigenvalues formulation of circulant matrix and we choose $c_0, c_1, \cdots, c_{n-2}, c_{n-1}$ is arithmetic progression (sequence).

2.2 Arithmetic Progression

Arithmetic progression or arithmetic sequence is a sequence of numbers so that the difference of the consecutive terms is constant. If the initial term of that sequence is denoted by u_0 and d is the common difference, then we get the successive members: $u_0, u_0 + d, u_0 + 2d, \cdots, u_0 + (j-1)d, \cdots$ which means that the n-th term is $u_n = u_0 + (n-1)d$ for all integers $n \geq 2$. The following proposition is easy to prove, and next, it will be referred to in the proof of the formulation of the eigenvalues. This proposition can be proved by mathematical induction.

Proposition 1. *In the arithmetic sequence, the sum of the first n terms is formulated as* $S_n = \frac{n(u_0+u_{n-1})}{2}$ *for any integer* $n \geq 2$. *Then, we have that the mean value of the series* S_n *can be formulated as* $\mu = \frac{S_n}{2} = \frac{(u_0+u_{n-1})}{2}$. *Furthermore, for the case of n is even, we have*

$$T_n = \sum_{j=0}^{n-i} (-1)^j u_j = \frac{-nd}{2}.$$

In the following, we define the circulant matrix with the entries in the first row having forming an arithmetic sequence which will become the main object of the topic in the subsequent discussion.

Definition 1. *Given constant values a and d and any integer* $n \geq 2$. *We define the* $n \times n$ *circulant matrix with the entry in the first row* $\{a + (j-1)d\}_{j=1}^{n}$ *is the matrix, denoted by* $A_{a,d,n}$, as

$$A_{a,d,n} = Circ(a, a+d, a+2d, \ldots, a+(n-2)d, a+(n-1)d).$$

3 A Theorem for Determinant and Inverse Formulation

For the proof of the following theorem, we refer to [15] as a basic theory.

Theorem 1. *Given constant values a and d and any integer* $n \geq 2$. *Let A be the circulant matrix* $A_{a,d,n}$, *and let* $\mu = \frac{2a+(n-1)d}{2}$ *be the mean value as stated in Proposition 1. If* $\mu \neq 0$, *then*

$$det\ (A) = \mu(-ad)^{n-1}$$

and

$$A^{-1} = \frac{1}{dn^2\mu} Circ(d - n\mu, d + n\mu, d, \cdots, d).$$

Proof. The following proof is described step by step in 5 steps. Let $A = A_{a,d,n} =$

$$\begin{pmatrix} a & a+d & a+2d & \cdots & a+(n-1)d \\ a+(n-1)d & a & a+d & \cdots & a+(n-2)d \\ a+(n-2)d & a+(n-1)d & a & \ddots & a+(n-3)d \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a+2d & a+3d & a+4d & \cdots & a+d \\ a+d & a+2d & a+3d & \cdots & a \end{pmatrix}$$

be the matrix as defined in Definition 1.

1. We apply E_1, a series of elementary row operations acting on $A = A_{a,d,n}$ by substracting the *i*-th row by the first row, for $i = 2,3,\cdots,n$. The result is

$$A \sim D_1 = \begin{pmatrix} a & a+d & a+2d & \cdots & a+(n-2)d & a+(n-1)d \\ (n-1)d & -d & -d & \cdots & -d & -d \\ (n-2)d & (n-2)d & -2d & \cdots & -2d & -2d \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2b & 2b & 2d & \cdots & -(n-2)d & -(n-2)d \\ d & d & d & \cdots & d & -(n-1)d \end{pmatrix}.$$

In this step, there exists a unique matrix $L_1 = E_1(I_n)$ such that $D_1 = L_1 A$ where

$$L_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & 1 & 0 \\ -1 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

2. Let K_1 be a series of elementary row operations row acting on D_1 by substracting the *j*th column by the first column, for $j = 2,3,\cdots,n$. The result is

$$A \sim D_1 = K_1(D_1) = \begin{pmatrix} a & d & 2d & \cdots & (n-2)d & (n-1)d \\ (n-1)d & -nd & -nd & \cdots & -nd & -nd \\ (n-2)d & 0 & -nd & \cdots & -2d & -nd \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2d & 0 & 0 & \cdots & -nd & -nd \\ d & 0 & 0 & \cdots & 0 & -nd \end{pmatrix}$$

and there exists matrix $R_1 = K_1(I_n)$ such that $D_2 = L_1 A R_1$, where

$$R_1 = \begin{pmatrix} 1 & -1 & -1 & \cdots & -1 & -1 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

3. Apply E_2, a series of elementary row operations acting on D_2 by substracting the *i*-th row by the $(i+1)$-th row, consecutively for $i = 2,3,\cdots,(n-1)$. The result is

$$A \sim D_3 = E_2(D_2) = \begin{pmatrix} a & d & 2d & \cdots & (n-2)d & (n-1)d \\ d & -nd & 0 & \cdots & 0 & 0 \\ d & 0 & -nd & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ d & 0 & 0 & \cdots & -nd & 0 \\ d & 0 & 0 & \cdots & 0 & -nd \end{pmatrix}$$

and there exists matrix $L_2 = E_2(L_1)$ such that $D_3 = L_2 A R_1$, where

$$L_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ -1 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

4. Let K_2 be a series of elementary row operations acting on D_3 by adding $\frac{1}{n}$ times the j-th column to the first column, for $j = 2, 3, \cdots, n$. The result is

$$A \sim D_4 = K_2(D_3) = \begin{pmatrix} T & d & 2d & \cdots & (n-2)d & (n-1)d \\ 0 & -nd & 0 & \cdots & 0 & 0 \\ 0 & 0 & -nd & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -nd & 0 \\ 0 & 0 & 0 & \cdots & 0 & -nd \end{pmatrix}$$

where

$$\begin{aligned} T &= a + \frac{d}{n} + \frac{2d}{n} + \cdots + \frac{(n-2)d}{n} + \frac{(n-1)d}{n} = a + \frac{d}{n}\sum_{i=1}^{n-1} i \\ &= a + \frac{dn(n-1)}{2n} = \frac{a + [a + d(n-1)]}{2} = \frac{2a + (n-1)d}{2} = \mu. \end{aligned}$$

Now, the determinant formulation is $\det(A) = \det(A_4) = \mu(-nd)^{n-1}$. Moreover, there exists matrix $R = K_2(R_1)$ such that $D_4 = L_2 A R$ where

$$R = \begin{pmatrix} U & -1 & -1 & \cdots & -1 & -1 \\ \frac{1}{n} & 1 & -1 & \cdots & 0 & 0 \\ \frac{1}{n} & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{n} & 0 & 0 & \cdots & 1 & -1 \\ -1 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

and in this case, $U = 1 - \sum_{i=1}^{n-1}\left(\frac{1}{n}\right) = 1 - \frac{n-1}{n} = \frac{1}{n}$.

5. We apply E_3, a series of elementary row operations acting on D_4 by adding $\frac{i-1}{n}$ times the i-th row to the first row, for $i = 2, 3, \cdots, n$. The result is a diagonal matrix

$$A \sim D = E_3(D_4) = \begin{pmatrix} \mu & 0 & 0 & \cdots & 0 & 0 \\ 0 & -nd & 0 & \cdots & 0 & 0 \\ 0 & 0 & -nd & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -nd & 0 \\ 0 & 0 & 0 & \cdots & 0 & -nd \end{pmatrix}$$

and there exists matrix $L = E_3(L_2)$ such that $D = LAR$ where

$$L = \begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ -1 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Finally, we obtain the formulation for the inverse

$$\begin{aligned} A^{-1} &= \mathrm{Circ}\left(\frac{1}{n^2\mu} - \frac{1}{nd}, \frac{1}{n^2\mu} + \frac{1}{nd}, \frac{1}{n^2\mu}, \cdots, \frac{1}{n^2\mu}\right) \\ &= \frac{1}{dn^2\mu}\mathrm{Circ}(d - n\mu, d + n\mu, d \cdots, d). \ \blacksquare \end{aligned}$$

The following corollary shows that $A_{a,d,n}$ is invertible if only if the first term is not equal to the negative of the last term in the arithmetic sequence of $A_{a,d,n}$.

Corollary 1. *The matix $A_{a,d,n}$ is invertible if only if $a \neq \frac{(1-n)d}{2}$.*

Proof. Based on the formulation of $A^{-1}_{a,d,n}$ in Theorem 1. $A_{a,d,n}$ is invertible if only if $dn^2\mu \neq 0 \Leftrightarrow \mu \neq 0 \Leftrightarrow 2a + (n-1)d \neq 0 \Leftrightarrow a \neq \frac{(1-n)d}{2}$. □

4 A Theorem for Eigenvalues Formulation

Recall the subgroup $S = \{1, \omega, \omega^2, \cdots, \omega^{n-1}\}$ in Section 2 which is a cyclic group, Gemetrically, all n elements in S are points on the complex plane. Those points occupy the unit circle and divide the circle into n equals parts. Thus, for $l = 1, 2, \cdots, \left\lfloor\frac{n}{2}\right\rfloor$ and $\theta = \frac{2\pi}{n}$, we have

$$\omega^l + \omega^{n-l} = \omega^l + \omega^{-l} = 2\cos(l\theta) \text{ and } \omega^l - \omega^{n-l} = \omega^l - \omega^{-l} = 2i\sin(l\theta) \qquad (4)$$

Furthermore, from the above fact, we have a lemma as follows.

Lemma 2. *Let* $m = \left\lfloor \frac{n-1}{2} \right\rfloor$, *then for* $k = 1, 2, \cdots, m$, *we have*

$$\sum_{t=1}^{m} cos\ (tk\theta) = \begin{cases} -\dfrac{1}{2} & when\ n\ is\ odd \\ \dfrac{-1 + (-1)^{k+1}}{2} & when\ n\ is\ even \end{cases}$$

Proof. Since $S = \{1, \omega, \omega^2, \cdots, \omega^{n-1}\}$ is cyclic group, then for $k = 1, 2, \cdots, m$, it is clear that T=$\{1, \omega^k, \omega^{2k}, \cdots, \omega^{(n-1)k}\}$ is a subgroup of S which also occupies the unit circle in the complex plane. Thus, we have

$$1 + \omega^k + \omega^{2k} + \cdots + \omega^{(n-1)k} = \frac{(\omega^k)^n - 1}{\omega^k - 1} = 0$$

And then using Equation 4, we obtain

$$\sum_{t=1}^{m} (\omega^{tk} + \omega^{-tk}) = \begin{cases} -1 & \text{when } n \text{ is odd} \\ -1 - \left(\omega^{\frac{n}{2}}\right)^k & \text{when } n \text{ is even} \end{cases} \Leftrightarrow$$

$$2\sum_{t=1}^{m} \cos(tk\theta) = \begin{cases} -1 & \text{when } n \text{ is odd} \\ -1 + (-1)^{k+1} & \text{when } n \text{ is even} \end{cases} \Leftrightarrow$$

$$\sum_{t=1}^{m} \cos(tk\theta) = \begin{cases} -\dfrac{1}{2} & \text{when } n \text{ is odd} \\ \dfrac{-1 + (-1)^{k+1}}{2} & \text{when } n \text{ is even.} \end{cases} \blacksquare$$

Theorem 3. *Given constant values a and d and any integer* $n \geq 2$. *Let A be the circulant matrix* $A = A_{a,d,n} = \text{Circ}(u_0, u_1, \cdots, u_{n-1})$, *and for* $j = 0, 1, 2, \cdots, n-1$, *let* λ_j *be the eigenvalues of A. If* $\theta = \frac{2\pi}{n}$ *and* $m = \left\lfloor \frac{n-1}{2} \right\rfloor$, *then* $\lambda_0 = \frac{n(2a+(n-1)d)}{3}$ *and for* $k = 1, 2, \cdots, m$, w*e obtain*

$$\lambda_k = R + C_k i \text{ and } \lambda_{n-k} = \overline{\lambda_k} = R - C_k i$$

where

$$R = \frac{-nd}{2} \text{ and } C_k = -d\sum_{t=1}^{m} (n - 2t)\sin(tk\theta).$$

We add $\lambda_{m+1} = \frac{-nd}{2}$ when n is even.

Proof. From Equation (2), consider first that in the context of matrix A, we have

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{-2} & \omega^{-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{-4} & \omega^{-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \omega^{-2} & \omega^{-4} & \cdots & \omega^4 & \omega^2 \\ 1 & \omega^{-1} & \omega^{-2} & \cdots & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n-2} \\ \lambda_{n-1} \end{pmatrix}$$

and so directly from Theorem 1, it clear that $\lambda_0 = \frac{n(u_0+u_{n-1})}{2} = \frac{n(2a+(n-1)d)}{2}$, and for the case of n is even, we also obtain that

$$\lambda_{m+1} = \lambda_{\frac{n}{2}} = \sum_{t=0}^{n} u_t\, \omega^{\frac{n}{2}t} = \sum_{t=0}^{n} (-1)^t\, u_t = \frac{-nd}{2}.$$

Next, for $k = 1, 2, \cdots, m$, notice that

$$\begin{aligned} \lambda_k + \lambda_{n-k} &= \sum_{t=0}^{n-1} u_t\, (\omega^{tk} + \omega^{t(n-k)}) = 2a + \sum_{t=1}^{n-1} u_t\, (\omega^{tk} + \omega^{-tk}) \\ &= 2a + \sum_{t=1}^{m} (u_t + u_{n-t})\, (\omega^{tk} + \omega^{-tk}) \end{aligned}$$

and applying Equation (4), we have

$$\begin{aligned} \lambda_k + \lambda_{n-k} &= 2a + 2\sum_{t=1}^{m} (2a + nd)\cos(\,tk\theta) \\ &= 2\left(a + (2a + nd)\sum_{t=1}^{m} \cos(\,tk\theta)\right) \end{aligned} \tag{5}$$

If n is even,

$$\begin{aligned} \lambda_k + \lambda_{n-k} &= 2a + \sum_{t=1}^{m} (u_t + u_{n-t})(\omega^{sk} + \omega^{-sk}) + 2(-1)^k u_{\frac{n}{2}} \lambda_k + \lambda_{n-k} \\ &= 2\left(a + (-1)^k(a + (m-1)d) + (2a + nd)\sum_{(t=1)}^{m} \cos(tk\theta)\right) \end{aligned} \tag{6}$$

Analogously, consider that

$$\begin{aligned} \lambda_k - \lambda_{n-k} &= \sum_{t=0}^{n-1} u_t\left(\omega^{tk} - \omega^{t(n-k)}\right) = \sum_{t=1}^{n-1} u_t\, (\omega^{tk} - \omega^{-tk}) \\ &= \sum_{t=1}^{m} (u_t + u_{n-t})(\omega^{tk} - \omega^{-tk}) \end{aligned}$$

Then applying Equation (4),

$$\lambda_k - \lambda_{n-k} = -2di \sum_{t=1}^{m} (n-2t)\sin(tk\theta). \tag{7}$$

By adding Equation (7) to (5) and subtracting Equation (7) from (5), we have

$$\lambda_k = R_k + iC_k$$
$$\lambda_{n-k} = R_k - iC_k$$

where

$$R_k = a + (2a+nd)\sum_{t=1}^{m}\cos(tk\theta)$$
$$C_k = -d\sum_{t=1}^{m}(n-2t)\sin(tk\theta)$$

Then applying Lemma 2, we obtain that $R_k = \frac{-nd}{2}$. If n is odd, by adding Equations (7) to (6) and applying Lemma 2, we also obtain

$$R_k = a + (-1)^k(a+(m+1)d) + (2a+nd)\left(-\frac{1}{2} + \frac{(-1)^{k+1}}{2}\right)$$
$$= -\frac{nd}{2} + (-1)^k\left(a+\frac{nd}{2}\right) + (-1)^{k+1}\left(a+\frac{nd}{2}\right)$$

This simplifies to $\frac{-nd}{2}$. □

5 Computational Remarks

In the following, we present a simple illustration of how to apply the formulations in Theorem 3. Then, by considering that illustration, we construct an efficient algorithm to compute the eigenvalues.

Example 1. (Illustration) For the matrix $A_{2,3,5}$ we have $a = 3, n = 5, m = 2,$ and $\theta = \frac{2\pi}{5}$. Then

$$\lambda_0 = \frac{(5(2\times 2+4\times 3))}{2} = 40,\ R = \frac{-5\times 3}{2} = -7.5$$

so that $\lambda_1 = -7.5 + C_1 i$ and $\lambda_4 = \overline{\lambda_1}$, $\lambda_2 = -7.5 + C_2 i$ and $\lambda_3 = \overline{\lambda_2}$ where

$$C_1 = -3\left(3\sin\frac{2\pi}{5} + \sin\frac{4\pi}{5}\right) \approx -10.32$$

$$C_2 = -3\left(3\sin\frac{4\pi}{5} + \sin\frac{8\pi}{5}\right) - 2\sqrt{3} \approx -2.44$$

For the matrix $A_{3,2,6}$, we have $a = 3, d = 2, n = 6, m = 2,\ and\ \theta = \frac{\pi}{3}$. Then

$$\lambda_0 = \frac{6(2\times3+5\times2)}{2} = 48, \lambda_3 = \frac{-6\times2}{2} = -6$$

Furthermore, $\lambda_1 = -6 + C_1 i$ and $\lambda_5 = \overline{\lambda_1},\ \lambda_2 = -6 + C_2 i$ and $\lambda_4 = \overline{\lambda_2}$ where

$$C_1 = -2\left(4\sin\frac{\pi}{3} + 2\sin\frac{2\pi}{3}\right) = -6\sqrt{3} \approx -10.39$$
$$C_2 = -2\left(4\sin\frac{2\pi}{3} + 2\sin\frac{4\pi}{3}\right) = -2\sqrt{3} \approx -3.46$$

From the above illustration, it is easy to verify that only $m = \left\lfloor\frac{n-1}{2}\right\rfloor$ eigenvalues are calculated iteratively, each of those iterative computations needs only m iterations. Besides, we can also see that all those computations did not involve any arithmetic of complex numbers. Thus, we can conclude that the computations of eigenvalues must be much faster than the computations based on the general formula as presented in Equation (1).

Algorithm 1.
INPUT : $A_{a,d,n}$ is the circulant matrix with arithmetic numbers.
OUTPUT : $\lambda_0, \lambda_1, \cdots, \lambda_{n-2}, \lambda_{n-1}$ are the eigenvalues of $A_{a,d,n}$.
1. $m \leftarrow \left\lfloor\frac{n-1}{2}\right\rfloor;\ \theta \leftarrow \frac{2\pi}{n}; \lambda_0 \leftarrow \frac{n(2n+(n-1)d)}{2};\ \ R \leftarrow \frac{-nd}{2};$
2. **if** $(n \bmod 2) = 0$ **then** $\lambda_{m+1} \leftarrow R$ **end if**;
3. **for** $k = 1$ **to** m **do**
    $C \leftarrow 0;\ S \leftarrow 0;\ T \leftarrow k\theta;$
    for $t = 1$ **to** m **do**
        $S \leftarrow S + T;\ s \leftarrow \sin S;\ \ y \leftarrow (n - 2t)s;\ C \leftarrow C + y;$
    end do;
    $C \leftarrow -dC;\ \ \lambda_k \leftarrow R + C.\mathrm{i};\ \ \lambda_{n-k} \leftarrow R - C.\mathrm{i};$
    end do;
4. **return**$(\ \lambda_0,\ \lambda_1, \cdots,\ \lambda_{n-2},\ \lambda_{n-1}\ )$.

6 Conclusion

The formulations for the eigenvalues, inverse, and determinant of a circulant matrix with arithmetic sequences are formulated in a simple way. The formulation method to derive the inverse and determinant is applying elementary operations of row or column to get a simpler equivalent matrix such that the formulations are easy to be derived. For the eigenvalues formulation, the previous result for the general case of circulant matrices can be simplified by exploiting the properties of the cyclic group induced by the set of all roots of $x^n - 1 = 0$ as the set of points in the unit circle in the complex plane, and also by considering the speciality of the arithmetic sequence. Then, we construct an algorithm for those eigenvalues formulation, and this algorithm shows a better computation compared to the previously known result for the general case of circulant matrices. We believe that the methods proposed in this article can be applied for any type of circulant matrices (such as left circulant, skew circulant or more general *r*-circulant) involving any type of sequence of numbers (such as Geometric, Harmonic, Pell, Lucas, Fibonacci, etc.) which become our research topics in the near future.

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