FURTHER EXPLORATION OF THE KLEE-MINTY PROBLEM.

BIB PARUHUM SILALAHI

Department of Mathematics,
Faculty of Mathematics and Natural Sciences,
Bogor Agricultural University
Jl. Meranti, Kampus IPB Darmaga, Bogor, 16680 Indonesia

Abstract. The Klee-Minty problem is explored in this paper. The coordinates formulas of all vertices of the Klee-Minty cube are presented. The subset representation of the vertices of the Klee-Minty cube is discussed. How to construct the Klee-Minty path is showed. It turns out that there are rich structures in the Klee-Minty path. We explore these structures.
Key words: Klee-Minty cube, Klee-Minty path, Klee-Minty problem.

1. Introduction

The Klee-Minty (KM) problem is a problem that had been presented by Klee and Minty in [3]. The n-dimensional KM problem is given by:

$$\begin{aligned} &\min \;\; y_n \\ &\text{subject to} \;\; \rho y_{k-1} \leq y_k \leq 1 - \rho y_{k-1}, \quad k = 1, \ldots, n, \end{aligned} \tag{1}$$

where ρ is small positive number by which the unit cube $[0,1]^n$ is squashed, and $y_0 = 0$. The domain (we denote as $\mathcal{C}^n$), which is called KM-cube, is a perturbation of the unit cube in $\mathbf{R}^n$. If $\rho = 0$ then the domain is the unit cube and for $\rho \in (0, \frac{1}{2})$ it is a perturbation of the unit cube which is contained in the unit cube itself. Since the perturbation is small, the domain has the same number of vertices as the unit cube, i.e. 2^n.

The KM-problem has become famous because Klee and Minty found a pivoting rule such that the simplex method requires $2^n - 1$ iterations to solve the problem (1).

$$y_{k-1} \leq y_k \leq 1 - y_{k-1}$$

In this paper, we explore the KM problem further. We provides formulas for the coordinates of all vertices of the KM cube, and discuss the subset representation of the vertices of the KM cube. Then we describe the KM path. We show that when using the subset representation, the KM path can easily be constructed by using the so-called *flipping* operation. It turns out that there are rich structures in the KM path. We explore these structures.

2. Vertices of the Klee-Minty cube

With the n-dimensional KM problem as defined in (1), we define the slack vectors $\underline{s}$ and $\bar{s}$ according to

$$\underline{s}_k = y_k - \rho y_{k-1}, \qquad k = 1, \ldots, n, \tag{2}$$

$$\bar{s}_k = 1 - y_k - \rho y_{k-1}, \quad k = 1, \ldots, n. \tag{3}$$

For any vertex of the KM cube we have either $\underline{s}_k = 0$ or $\bar{s}_k = 0$, for each k. As a consequence, each vertex can be characterized by the subset of the index set $\mathcal{I}= \{1, 2, \ldots, n\}$ consisting of the indices k for which $\bar{s}_k$ vanishes (and hence $\underline{s}_k$ is positive). Therefore, given a vertex v we define

$$S_v = \{k \,:\, \bar{s}_k = 0\} \equiv \{k \,:\, \underline{s}_k > 0\}.$$

Note that the KM cube has 2^n vertices. Since 2^n is also the number of subsets of the index set $\mathcal{I}$, each subset S of the index set uniquely determines a vertex. We denote this vertex as v^S. Given S, the coordinates of v^S in the y-space can easily be solved from (2) and (3), because we then have $\bar{s}_k = 0$ if $k \in S$ and $\underline{s}_k = 0$ if $k \notin S$, which yields n equations in the entries of the vector y. When defining $y_0 = 0$ and $y = v^S$, one easily deduces that

$$y_k = \begin{cases} 1 - \rho y_{k-1}, & k \in S, \\ \rho y_{k-1}, & k \notin S. \end{cases} \tag{4}$$

Since $y_0 = 0$, we have $y_1 \in \{0, 1\}$. This together with (4) implies that y_k is a polynomial in ρ whose degree is at most $k - 1$. Moreover, the coefficients of this polynomial take only the values 0, 1 and -1, and the nonzero coefficients alternate between 1 and -1. Finally if $y_k \neq 0$ then the lowest degree term has coefficient 1.

We can be more specific. Let

$$S = \{s_1,\, s_2,\, \ldots,\, s_m\}, \quad s_0 = 0 < s_1 < s_2 < \ldots < s_m < s_{m+1} := n+1.$$

Then the entries of y are given by the following lemma. In this lemma we define an empty sum to be equal to zero.

Lemma 1. *One has*

$$y_{s_i} = \sum_{j=1}^{i} (-1)^{i+j} \rho^{s_i - s_j}, \quad 0 \leq i \leq m, \tag{5}$$

and

$$y_k = \rho^{k - s_i} y_{s_i}, \quad s_i < k < s_{i+1}, \quad k \notin S. \tag{6}$$

Proof. If $k \notin S$ then the definition of S implies that $s_i < k < s_{i+1}$ for some i, with $0 \leq i \leq m$. It follows from (4) that in that case

$$y_k = \rho y_{k-1} = \ldots = \rho^{k-s_i-1} y_{s_i+1} = \rho^{k-s_i} y_{s_i},$$

proving (6).

So it remains to prove (5). The proof uses induction with respect to the index i in (5). Before entering this proof it may be worth noting that (5) expresses y_{s_i} as a polynomial in ρ of degree $s_i - s_1$. The lowest degree term occurs for $j = i$, and hence this term equals $(-1)^{2i}\rho^0 = 1$.

For $i = 0$ the sum in (5) becomes empty, whence we obtain $y_0 = 0$, as it should. This proves that (5) holds if $i = 0$. Now assume that (5) holds for some i, with $0 \leq i < m$. Since $s_{i+1} \in S$, according to (4) we have

$$y_{s_{i+1}} = 1 - \rho y_{s_{i+1}-1}. \tag{7}$$

At this stage we need to distinguish two cases: $s_{i+1} - 1 \in S$ (case I) and $s_{i+1} - 1 \notin S$ (case II).

In case I we must have $s_{i+1} - 1 = s_i$. Since (5) holds for y_{s_i} it follows from (7) that

$$\begin{aligned} y_{s_{i+1}} = 1 - \rho y_{s_i} &= 1 - \rho \textstyle\sum_{j=1}^{i} (-1)^{i+j} \rho^{s_i - s_j} \\ &= 1 - \textstyle\sum_{j=1}^{i} (-1)^{i+j} \rho^{s_i + 1 - s_j} \\ &= 1 + \textstyle\sum_{j=1}^{i} (-1)^{i+1+j} \rho^{s_{i+1} - s_j}. \end{aligned}$$

In case II we may use (6), which gives

$$\begin{aligned} y_{s_{i+1}-1} &= \rho^{s_{i+1}-1-s_i} y_{s_i} = \rho^{s_{i+1}-1-s_i} \textstyle\sum_{j=1}^{i} (-1)^{i+j} \rho^{s_i - s_j} \\ &= \textstyle\sum_{j=1}^{i} (-1)^{i+j} \rho^{s_{i+1}-1-s_j}, \end{aligned}$$

whence (7) yields that

$$y_{s_{i+1}} = 1 - \rho \sum_{j=1}^{i} (-1)^{i+j} \rho^{s_{i+1}-1-s_j} = 1 + \sum_{j=1}^{i} (-1)^{i+1+j} \rho^{s_{i+1}-s_j}.$$

We conclude that in both cases we have

$$y_{s_{i+1}} = 1 + \sum_{j=1}^{i} (-1)^{i+1+j} \rho^{s_{i+1}-s_j} = \sum_{j=1}^{i+1} (-1)^{i+1+j} \rho^{s_{i+1}-s_j},$$

which completes the proof. □

3. The Klee-Minty path

As already mentioned previously, Klee and Minty found a pivoting rule such that the simplex method requires $2^n - 1$ iterations to solve the problem (1). This implies that the method passes through all vertices of the KM cube before finding the optimal vertex (which is of course the zero vector). We call this path along all vertices the KM path. It is well known in which order the vertices are visited. This can be easily described by using the subset representation of the vertices introduced in the previous section. The path starts at vertex $(0, \ldots, 0, 1)$ whose subset is the subset $S_1 = \{n\}$. The subsequent subsets are obtained by an operation which we call *flipping* an index with respect to a subset S [4]. Given a subset S and an index i, flipping i (with respect to S) means that we add i to S if $i \notin S$, and remove i from S if $i \in S$. Now let S_k denote the subset corresponding to the k-th vertex on the KM path. Then S_{k+1} is obtained from S_k as follows:

- if $|S_k|$ is odd, then flip 1;
- if $|S_k|$ is even, then flip the element following the smallest element in S_k.

Denoting the resulting sequence as P_n, we can now easily construct the KM path for small values of n:

$$
\begin{aligned}
&P_1: \ \{1\} \to \emptyset \\
&P_2: \ \{2\} \to \{1,2\} \to \{1\} \to \emptyset \\
&P_3: \ \{3\} \to \{1,3\} \to \{1,2,3\} \to \{2,3\} \to P_2 \\
&P_4: \ \{4\} \to \{1,4\} \to \{1,2,4\} \to \{2,4\} \to \{2,3,4\} \to \{1,2,3,4\} \\
&\qquad \to \{1,3,4\} \to \{3,4\} \to P_3.
\end{aligned}
$$

Table 1 shows the subsets and the corresponding vectors y for the KM path for $n = 4$. The corresponding tables for $n = 2$ and $n = 3$ are subtables, as indicated. Note that if subsets S and S' differ only in n, and $y = v_S$ and $y' = v_{S'}$, then we have

$$y_i = y'_i, \quad 1 \le i < n, \quad y_n + y'_n = 1.$$

Obviously, the subsets of two subsequent vertices differ only in one element. For the corresponding subsets, S_k and S_{k+1} say, we denote this element by i_k. Then we have $\bar{s}_{i_k} = 0$ in one of these vertices, and in the other vertex $\bar{s}_{i_k} > 0$, or equivalently $\underline{s}_{i_k} = 0$. On the interior of the edge connecting these two vertices we will have $\underline{s}_{i_k} > 0$ and $\bar{s}_{i_k} > 0$. If $n = 4$ then, when following the KM path, the flipping index i_k runs through the following sequence:

$$1,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1,\ 4,\ 1,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1.$$

So if $n = 4$ then the flipping index 8 times equals 1, 4 times 2, 2 times 3, and once 4.

Table 2 shows the slack vector $\underline{s}$ and Table 3 the slack vector $\bar{s}$ for each of the vertices. For a graphical illustration (with $n = 3$) we refer to Figure 1.

4. Sequence of vertices in the Klee-Minty path

One easily observes that for $n \in \{2, 3, 4\}$, the second half of P_n is just P_{n-1} whereas the first half of P_n arises by reversing the order of the sequence P_{n-1} and adding the element n to each sets in the resulting

S	y_1	y_2	y_3	y_4
$\{4\}$	0	0	0	1
$\{1,4\}$	1	ρ	ρ^2	$1-\rho^3$
$\{1,2,4\}$	1	$1-\rho$	$\rho-\rho^2$	$1-\rho^2+\rho^3$
$\{2,4\}$	0	1	ρ	$1-\rho^2$
$\{2,3,4\}$	0	1	$1-\rho$	$1-\rho+\rho^2$
$\{1,2,3,4\}$	1	$1-\rho$	$1-\rho+\rho^2$	$1-\rho+\rho^2-\rho^3$
$\{1,3,4\}$	1	ρ	$1-\rho^2$	$1-\rho+\rho^3$
$\{3,4\}$	0	0	1	$1-\rho$
$\{3\}$	0	0	1	ρ
$\{1,3\}$	1	ρ	$1-\rho^2$	$\rho-\rho^3$
$\{1,2,3\}$	1	$1-\rho$	$1-\rho+\rho^2$	$\rho-\rho^2+\rho^3$
$\{2,3\}$	0	1	$1-\rho$	$\rho-\rho^2$
$\{2\}$	0	1	ρ	ρ^2
$\{1,2\}$	1	$1-\rho$	$\rho-\rho^2$	$\rho^2-\rho^3$
$\{1\}$	1	ρ	ρ^2	ρ^3
$\emptyset$	0	0	0	0

TABLE 1. The KM path (in the y-space) for $n = 4$.

sequence. Hence, when denoting the first half of P_n as $\bar{P}^n_{n-1}$ we have for $n \in \{2,3,4\}$ that $P_n = \bar{P}^n_{n-1} \to P_{n-1}$. Indeed, when defining $P_0 = \emptyset$ then this pattern holds for each $n \geq 1$, as stated in the following lemma.

Lemma 2. *For $n \geq 1$, one has*

$$P_n \;:\; \bar{P}^n_{n-1} \to P_{n-1}. \tag{8}$$

Proof. The proof uses induction with respect to n. We already know that the lemma holds if $n \leq 4$. Therefore, (8) holds if $n = 1$. Suppose that $n \geq 2$. Let S_k denote the k-th subset in the sequence P_{n-1}. By the induction hypothesis we have $S_1 = \{n-1\}$ and $S_{2^{n-1}} = \emptyset$. The first set in P_n is the set $\{n\} = \{n\} \cup S_{2^{n-1}}$. For $2 \leq k \leq 2^{n-1}$, we consider the set $S = S_k \cup \{n\}$ and we show below that its successor is the set $S_{k-1} \cup \{n\}$. This will imply that the 2^{n-1}-th set in P_n is $S_1 \cup \{n\} = \{n-1, n\}$, whose successor is the set $\{n-1\}$, the first set of P_{n-1}. This makes clear that it suffices for the proof of the lemma if we show that for each set S_k in P_{n-1} the successor of $S_k \cup \{n\}$ is the set $S_{k-1} \cup \{n\}$. This can be shown as follows.

If $|S|$ is even then the successor of S in P_n arises by flipping the element following the smallest element in S. If this smallest element equals $n-1$ then we must have $S = \{n-1, n\}$, and then the successor

S	$\underline{s}_1$	$\underline{s}_2$	$\underline{s}_3$	$\underline{s}_4$
$\{4\}$	0	0	0	1
$\{1,4\}$	1	0	0	$1-2\rho^3$
$\{1,2,4\}$	1	$1-2\rho$	0	$1-2\rho^2+2\rho^3$
$\{2,4\}$	0	1	0	$1-2\rho^2$
$\{2,3,4\}$	0	1	$1-2\rho$	$1-2\rho+2\rho^2$
$\{1,2,3,4\}$	1	$1-2\rho$	$1-2\rho+2\rho^2$	$1-2\rho+2\rho^2-2\rho^3$
$\{1,3,4\}$	1	0	$1-2\rho^2$	$1-2\rho+2\rho^3$
$\{3,4\}$	0	0	1	$1-2\rho$
$\{3\}$	0	0	1	0
$\{1,3\}$	1	0	$1-2\rho^2$	0
$\{1,2,3\}$	1	$1-2\rho$	$1-2\rho+2\rho^2$	0
$\{2,3\}$	0	1	$1-2\rho$	0
$\{2\}$	0	1	0	0
$\{1,2\}$	1	$1-2\rho$	0	0
$\{1\}$	1	0	0	0
$\emptyset$	0	0	0	0

TABLE 2. The KM path (in the $\underline{s}$-space) for $n = 4$.

of S is the set $\{n-1\} = S_1$, which is the first element of P_{n-1}. Otherwise the smallest element is at most $n-2$, and then, since $|S_k|$ is odd, the successor of S is equal to $S_{k-1} \cup \{n\}$. The latter follows since S and S_{k-1} have the same smallest element and $|S_{k-1}|$ is even.

If $|S|$ is odd then the successor of S in P_n arises by flipping 1. Since $|S_k|$ is even flipping 1 yields the predecessor of S_k in P_{n-1}, which is S_{k-1}. Hence we find again that the successor of S is $S_{k-1} \cup \{n\}$. This completes the proof. □

According to this lemma, the index i that occurs K times as flipping index in P_{n-1} will occur $2K$ times in P_n, i.e. K times in $\bar{P}^n_{n-1}$ and K times in P_{n-1}. The index n flips only at the last set in $\bar{P}^n_{n-1}$, which gives the first set in P_{n-1}. These sets are $\{n-1, n\} \setminus \{0\}$ and $\{n-1\} \setminus \{0\}$ respectively. The complement operation of $\{0\}$ is applied to adjust for the case where $n = 1$. As an immediate consequence we have the following corollary.

Corollary 1. *The index i, $1 \le i \le n$, never flips in P_k, for $0 \le k < i$. It flips for the first time in P_i when applied to the set $\{i-1, i\} \setminus \{0\}$, which yields the set $\{i-1\} \setminus \{0\}$.*

From Lemma 2, for $0 \le i < n$, we can obtain

$$P_n \ : \ \bar{P}^n_{n-1} \to \bar{P}^{n-1}_{n-2} \to \ldots \to \bar{P}^{i+1}_i \to P_i. \tag{9}$$

S	$\bar{s}_1$	$\bar{s}_2$	$\bar{s}_3$	$\bar{s}_4$
$\{4\}$	1	1	1	0
$\{1,4\}$	0	$1-2\rho$	$1-2\rho^2$	0
$\{1,2,4\}$	0	0	$1-2\rho+2\rho^2$	0
$\{2,4\}$	1	0	$1-2\rho$	0
$\{2,3,4\}$	1	0	0	0
$\{1,2,3,4\}$	0	0	0	0
$\{1,3,4\}$	0	$1-2\rho$	0	0
$\{3,4\}$	1	1	0	0
$\{3\}$	1	1	0	$1-2\rho$
$\{1,3\}$	0	$1-2\rho$	0	$1-2\rho+2\rho^3$
$\{1,2,3\}$	0	0	0	$1-2\rho+2\rho^2-2\rho^3$
$\{2,3\}$	1	0	0	$1-2\rho+2\rho^2$
$\{2\}$	1	0	$1-2\rho$	$1-2\rho^2$
$\{1,2\}$	0	0	$1-2\rho+2\rho^2$	$1-2\rho^2+2\rho^3$
$\{1\}$	0	$1-2\rho$	$1-2\rho^2$	$1-2\rho^3$
$\emptyset$	1	1	1	1

TABLE 3. The KM path (in the $\bar{s}$-space) for $n = 4$.

According to Corollary 1, index i is flipped for the first time in P_i, hence we have the next corollary.

Corollary 2. *The index i flips for the last time in P_n at the set $\{i-1, i\} \setminus \{0\}$, which yields the set $\{i-1\} \setminus \{0\}$.*

The sequence $\bar{P}^n_{n-1}$ is equal to the sequence which is obtained by reversing the sequence

$$\bar{P}^{n-1}_{n-2} \to \ldots \to \bar{P}^{i+1}_i \to P_i$$

and by adding the element n to each sets in the resulting sequence. Thus the next corollary follows.

Corollary 3. *The index i flips for the first time in P_n at the set $\{i-1, n\} \setminus \{0\}$, which yields the set $\{i-1, i, n\} \setminus \{0\}$.*

Moreover, by letting J_i be any subset of $\mathcal{I} \setminus \{1, \ldots, i\}$, we have the following corollary.

Corollary 4. *The index i flips in P_n when it is applied to either*

$$\{i-1, i\} \setminus \{0\} \cup J_i \quad \text{or} \quad \{i-1\} \setminus \{0\} \cup J_i.$$

Proof. Let us consider P_n as in (9). The flipping indexes of two sets that connecting two sequences of sets consecutively are n, $n-1$, $\ldots$, $i+1$. In

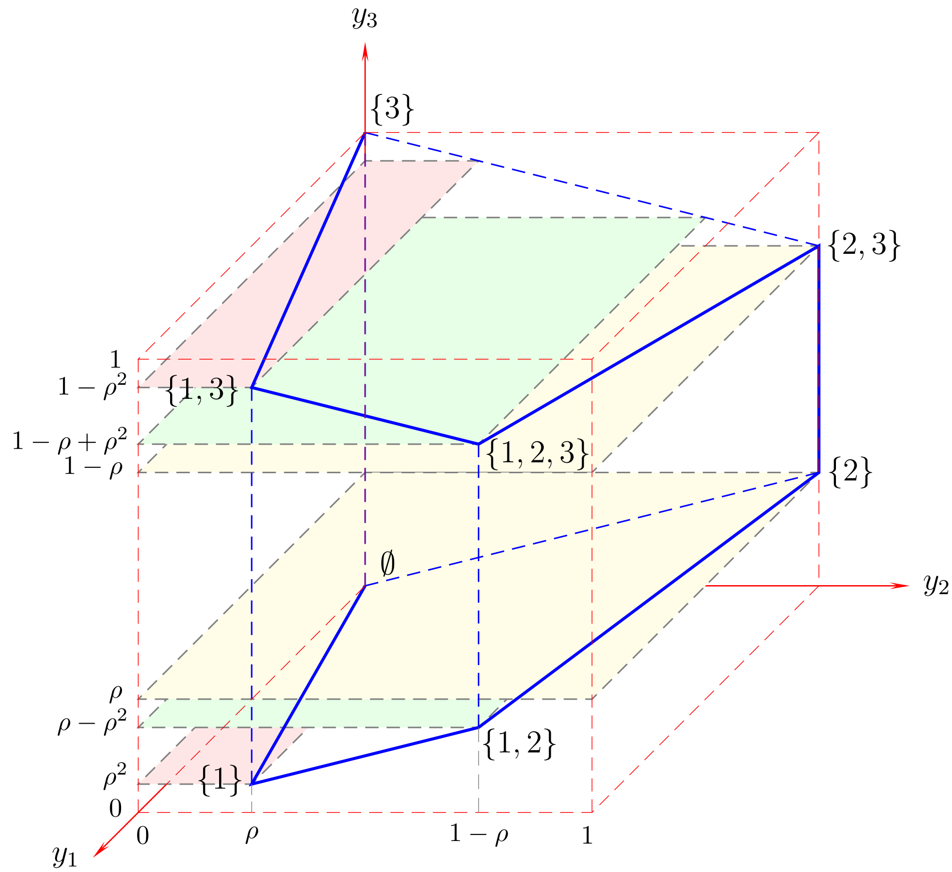

FIGURE 1. Unit cube (red dashed), KM cube (blue dashed) and KM path (blue solid) for $n = 3$.

this case there is no index which is equal to i. In P_i, flipping an index i is only applied to the set $\{i-1, i\}\setminus\{0\}$ which yield the set $\{i-1\}\setminus\{0\}$. Let us call the pair of two sets, where an index i is flipped with respect to one of the sets and another one is its successor, as pair of sets with flipping index i. Generally, the pairs of sets with flipping index i that appear in $\bar{P}_k^{k+1}$, $k \geq i$, definitely equal to pairs of sets which is resulted from the union of each set of pair of sets with flipping index i in P_k by $\{k+1\}$. By taking J_i as any subset of $\mathcal{I}\setminus\{1, \ldots, i\}$, we obtain that the pairs of sets with flipping index i in P_n are $\{i-1, i\}\setminus\{0\}\cup J_i$ and $\{i-1\}\setminus\{0\}\cup J_i$. This implies the corollary. □

The following corollary expresses how many times the index i is flipped in P_n . We have discussed this number for small value of n previously.

Lemma 3. *The index i, $1 \leq i \leq n$, is flipped in P_n exactly*

$$2^{n-i} \textit{ times.}$$

Proof. The index n is flipped only 1 time in P_n. By using the recursive pattern of P_n as in Lemma 2 and Corollary 1, in P_n, the index

$$\begin{cases} n-1\ , & \text{is flipped 2 times,} \\ n-2\ , & \text{is flipped 4 times,} \\ \quad\vdots & \\ n-k\ , & \text{is flipped } 2^k \text{ times,} \\ \quad\vdots & \\ \quad 1 \quad , & \text{is flipped } 2^{n-1} \text{ times.} \end{cases}$$

The lemma follows by taking $i = n - k$. □

When n is given, the set S_k is uniquely determined by k, and vice versa. Moreover, for each k $(2 \leq k \leq 2^n)$, the sets S_{k-1} and S_k differ only in one element. This means that the sequence P_n defines a so-called *Gray code*. Such codes have been studied thoroughly, also because of their many applications.[1] It may be worth mentioning some results from the literature that make the one-to-one correspondence between k and S_k more explicit. The next proposition is the main result (Theorem 6(ii)) in [1].[2]

Proposition 1. *One has $i \in S_k$ if and only if*

$$\left\lfloor \frac{2^n - k}{2^i} + \frac{1}{2} \right\rfloor \bmod 2 = 1. \tag{10}$$

Given k, by computing the left-hand side expression in (10) for $i = 1, 2, \dots, n$ we get the set S_k. Conversely, when S_k is given we can find k (also in n iterations) by using Corollary 24 in [1]. This goes as follows. We first form the binary representation $b_n \dots b_2 b_1$ of S_k, with $b_i = 1$ if $i \in S_k$ and $b_i = 0$ otherwise. We then replace b_i by 0 if the number of 1's in $b_n \dots b_2 b_1$ to the left of b_i (including b_i itself) is even, and by 1 if this number is odd. The resulting binary n-word $a_n \dots a_2 a_1$ is the binary representation of some natural number, let it be K. Then $k = 2^n - K$. For example, let $S_k = \{2, 3\}$ and $n = 4$. Then

$$S_k \equiv b_n \dots b_2 b_1 = 0110 \to a_n \dots a_2 a_1 = 0100 \equiv 4 \to k = 2^4 - 4 = 12,$$

which is in accordance with Table 1.

---

[1] Gray codes were first designed to speed up telegraphy, but now have numerous applications such as in addressing microprocessors, hashing algorithms, distributed systems, detecting/correcting channel noise and in solving problems such as the Towers of Hanoi, Chinese Ring and Brain and Spinout.

[2] It simplifies an earlier result in [2], namely

$$i \in S_k \Leftrightarrow \binom{2^n - 2^{i-1} - 1}{\lfloor 2^n - 2^{i-2} - \frac{2^n+1-k}{2} \rfloor} \bmod 2 = 1.$$

5. Edges of the Klee-Minty path

An edge of the KM path in $\mathcal{C}^n$ is a line segment connecting two consecutive vertices of the KM path. As before, we represent the k-th vertex on the KM path by the set S_k. The edge connecting S_k and S_{k+1} is denoted as e_k^n. The set of all edges of the n-dimensional KM path, denoted by E^n, is therefore given by

$$E^n = \bigcup_{k=1}^{2^n-1} e_k^n. \tag{11}$$

Remember that S_k and S_{k+1} differ only in one element, which we denote as i_k. On the interior of the edge connecting these two vertices we have $\underline{s}_{i_k} > 0$ and $\bar{s}_{i_k} > 0$. For any $j \neq i_k$, we have $\bar{s}_j = 0$ if $j \in S_k \cap S_{k+1}$ and $\underline{s}_j = 0$ otherwise. So it follows that if $j \neq i_k$ then either $\underline{s}_j = 0$ on e_k^n or $\bar{s}_j = 0$ on e_k^n .

For $i_k \neq 1$, we have either $\underline{s}_1 = 0$ or $\bar{s}_1 = 0$, which implies either $y_1 = 0$ or $y_1 = 1$ on e_k^n. Since for any $j < i_k$ we have either $\underline{s}_j = 0$ or $\bar{s}_j = 0$ on e_k^n, we may conclude that y_j is constant on e_k^n. Summarizing, we may state that on e_k^n we have the following properties:

(*i*) $\underline{s}_{i_k} > 0$ or $\bar{s}_{i_k} > 0$,
(*ii*) $j \neq i_k \ : \ \bar{s}_j = 0$ or $\underline{s}_j = 0$,
(*iii*) $1 \leq j < i_k \ : \ y_j$ is constant.

Table 4 and Table 5 shows the slack vector $\underline{s}$ and $\bar{s}$ on e_k^n for $n = 4$. The subtables show $\underline{s}$ and $\bar{s}$ for $n = 1, n = 2$ and $n = 3$.

We therefore can describe the edge e_k^n as follows

$e_k^n = \{y \in \mathcal{C}^n \ : \ \text{ for any } j \neq i_k, \ \bar{s}_j = 0 \text{ if } j \in S_k \cap S_{k+1}, \ \underline{s}_j = 0 \text{ otherwise}\}.$

Or, in other words, since

$$S_k \cup S_{k+1} = (S_k \cap S_{k+1}) \cup \{i_k\}, \qquad 1 \leq k < 2^n,$$

we may write

$$e_k^n = \left\{ \begin{array}{l} y \in \mathcal{C}^n \ : \\ \bar{s}_j = 0 \text{ if } j \in S_k \cap S_{k+1}, \\ \underline{s}_j = 0 \text{ if } j \notin S_k \cup S_{k+1} \end{array} \right\}. \tag{12}$$

Further elaborating (*iii*) we get the following lemma.

Lemma 4. *Let i_k be the flipping element for S_k, then on e_k^n for $1 \leq j < i_k$ one has*

$$y_j = \begin{cases} 1 \ , \ j = i_k - 1, \\ 0 \ , \ \textit{otherwise.} \end{cases}$$

Edge	$\underline{s}_1$	$\underline{s}_2$	$\underline{s}_3$	$\underline{s}_4$
$\{4\} - \{1,4\}$	$(0,1)$	0	0	$(1-2\rho^3, 1)$
$\{1,4\} - \{1,2,4\}$	1	$(0, 1-2\rho)$	0	$(1-2\rho^2+2\rho^3, 1-2\rho^3)$
$\{1,2,4\} - \{2,4\}$	$(0,1)$	$(1-2\rho, 1)$	0	$(1-2\rho^2, 1-2\rho^2+2\rho^3)$
$\{2,4\} - \{2,3,4\}$	0	1	$(0, 1-2\rho)$	$(1-2\rho+2\rho^2, 1-2\rho^2)$
$\{2,3,4\} - \{1,2,3,4\}$	$(0,1)$	$(1-2\rho, 1)$	$(1-2\rho, 1-2\rho+2\rho^2)$	$(1-2\rho+2\rho^2-2\rho^3, 1-2\rho+2\rho^2)$
$\{1,2,3,4\} - \{1,3,4\}$	1	$(0, 1-2\rho)$	$(1-2\rho+2\rho^2, 1-2\rho^2)$	$(1-2\rho+2\rho^3, 1-2\rho+2\rho^2-2\rho^3)$
$\{1,3,4\} - \{3,4\}$	$(0,1)$	0	$(1-2\rho^2, 1)$	$(1-2\rho, 1-2\rho+2\rho^3)$
$\{3,4\} - \{3\}$	0	0	1	$(0, 1-2\rho)$
$\{3\} - \{1,3\}$	$(0,1)$	0	$(1-2\rho^2, 1)$	0
$\{1,3\} - \{1,2,3\}$	1	$(0, 1-2\rho)$	$(1-2\rho+2\rho^2, 1-2\rho^2)$	0
$\{1,2,3\} - \{2,3\}$	$(0,1)$	$(1-2\rho, 1)$	$(1-2\rho, 1-2\rho+2\rho^2)$	0
$\{2,3\} - \{2\}$	0	1	$(0, 1-2\rho)$	0
$\{2\} - \{1,2\}$	$(0,1)$	$(1-2\rho, 1)$	0	0
$\{1,2\} - \{1\}$	1	$(0, 1-2\rho)$	0	0
$\{1\} - \emptyset$	$(0,1)$	0	0	0

TABLE 4. Edges of the KM path (in $\underline{\text{s}}$-space) for $n = 4$.

Edge	$\bar{s}_1$	$\bar{s}_2$	$\bar{s}_3$	$\bar{s}_4$
$\{4\} - \{1,4\}$	$(0,1)$	$(1-2\rho,1)$	$(1-2\rho^2,1)$	0
$\{1,4\} - \{1,2,4\}$	0	$(0,1-2\rho)$	$(1-2\rho+2\rho^2,1-2\rho^2)$	0
$\{1,2,4\} - \{2,4\}$	$(0,1)$	0	$(1-2\rho,1-2\rho+2\rho^2)$	0
$\{2,4\} - \{2,3,4\}$	1	0	$(0,1-2\rho)$	0
$\{2,3,4\} - \{1,2,3,4\}$	$(0,1)$	0	0	0
$\{1,2,3,4\} - \{1,3,4\}$	0	$(0,1-2\rho)$	0	0
$\{1,3,4\} - \{3,4\}$	$(0,1)$	$(1-2\rho,1)$	0	0
$\{3,4\} - \{3\}$	1	1	0	$(0,1-2\rho)$
$\{3\} - \{1,3\}$	$(0,1)$	$(1-2\rho,1)$	0	$(1-2\rho,1-2\rho+2\rho^3)$
$\{1,3\} - \{1,2,3\}$	0	$(0,1-2\rho)$	0	$(1-2\rho+2\rho^3,1-2\rho+2\rho^2-2\rho^3)$
$\{1,2,3\} - \{2,3\}$	$(0,1)$	0	0	$(1-2\rho+2\rho^2-2\rho^3,1-2\rho+2\rho^2)$
$\{2,3\} - \{2\}$	1	0	$(0,1-2\rho)$	$(1-2\rho+2\rho^2,1-2\rho^2)$
$\{2\} - \{1,2\}$	$(0,1)$	0	$(1-2\rho,1-2\rho+2\rho^2)$	$(1-2\rho^2,1-2\rho^2+2\rho^3)$
$\{1,2\} - \{1\}$	0	$(0,1-2\rho)$	$(1-2\rho+2\rho^2,1-2\rho^2)$	$(1-2\rho^2+2\rho^3,1-2\rho^3)$
$\{1\} - \emptyset$	$(0,1)$	$(1-2\rho,1)$	$(1-2\rho^2,1)$	$(1-2\rho^3,1)$

TABLE 5. Edges of the KM path (in $\bar{s}$-space) for $n = 4$.

Proof. The only different element in S_k and S_{k+1} definitely is the index i_k that we flip in the set S_k. According to Corollary 4, the only different element i_k happens in pair of sets $\{i_k - 1, i_k\} \setminus \{0\} \cup J_{i_k}$ and $\{i_k - 1\} \setminus \{0\} \cup J_{i_k}$, where J_{i_k} is any subset of $\mathcal{I} \setminus \{1, \ldots, i_k\}$. This means that on the edge connecting S_k and S_{k+1} we have

$$\underline{s}_1 = 0,\ \underline{s}_2 = 0,\ \ldots,\ \underline{s}_{i_k-2} = 0,\ \bar{s}_{i_k-1} = 0.$$

From $\underline{s}_1 = 0$ we get $y_1 = 0$. Then subsequently we get $y_2 = 0, \ldots, y_{i_k-2} = 0$, $y_{i_k-1} = 1$ from $\underline{s}_2 = 0, \ldots, \underline{s}_{i_k-2} = 0$, $\bar{s}_{i_k-1} = 0$, which proves the lemma. □

One may use Table 1 to verify Lemma 4 for $n \leq 4$.

References

[1] M.W. Bunder, K.P. Tognetti, and G.E. Wheeler. On binary reflected Gray codes and functions. *Discrete Mathematics*, 308:1690–1700, 2008.

[2] M. Conder. Explicit definition of the binary reflected Gray codes. *Discrete Mathematics*, 195:245–249, 1999.

[3] V. Klee and G.J. Minty. How good is the simplex algorithm? In *Inequalities, III (Proc. Third Sympos., Univ. California, Los Angeles, Calif., 1969; dedicated to the memory of Theodore S. Motzkin)*, pages 159–175. Academic Press, New York, 1972.

[4] C. Roos. An exponential example for Terlaky's pivoting rule for the criss–cross simplex method. *Mathematical Programming*, 46:79–84, 1990.