

CONSISTENCY OF A UNIFORM KERNEL ESTIMATOR FOR INTENSITY OF A PERIODIC POISSON PROCESS WITH UNKNOWN PERIOD

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ABSTRACT. A uniform kernel estimator for intensity of a periodic Poisson process with unknown period is presented and a proof of its consistency is discussed. The result presented in this paper is a special case of that in [3]. The aim of discussing a uniform kernel estimator is in order to be able to present a relatively simpler proof of consistency compared to that in [3]. This is a joint work with R. Helmers and R. Zitikis.

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1. INTRODUCTION AND MAIN RESULT

In this paper, a uniform kernel estimator for intensity of a periodic Poisson process with unknown period is presented and a proof of its consistency is discussed. The result presented here is a special case of that in [3] and chapter 3 of [5].

Let X be a Poisson process on $[0, \infty)$ with (unknown) locally integrable intensity function λ . We assume that λ is a periodic function with unknown period τ . We do not assume any parametric form of λ , except that it is periodic. That is, for each point $s \in [0, \infty)$ and all $k \in \mathbf{Z}$, with \mathbf{Z} denotes the set of integers, we have

$$\lambda(s + k\tau) = \lambda(s). \quad (1.1)$$

Suppose that, for some $\omega \in \Omega$, a single realization $X(\omega)$ of the Poisson process X defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with intensity function λ is observed, though only within a bounded interval $[0, n]$. Our goal in this paper is: (a) To study construction of a uniform kernel estimator for λ at a given point $s \in [0, n]$ using only a single realization $X(\omega)$ of the Poisson process X observed in interval $[0, n]$. (The

requirement $s \in [0, n]$ can be dropped if we know the period τ .) (b) To determine the minimal conditions for having weak convergence of this estimator.

Note that, since λ is a periodic function with period τ , the problem of estimating λ at a given point $s \in [0, n]$ can be reduced into a problem of estimating λ at a given point $s \in [0, \tau)$. Hence, for the rest of this paper, we assume that $s \in [0, \tau)$.

Note also that, the meaning of the asymptotic $n \rightarrow \infty$ in this paper is somewhat different from the classical one. Here n does not denote our sample size, but it denotes the length of the interval of observations. The size of our samples is a random variable denoted by $X([0, n])$.

Let $\hat{\tau}_n$ be any consistent estimator of the period τ , that is, $\hat{\tau}_n \xrightarrow{p} \tau$, as $n \rightarrow \infty$. For example, one may use the estimators constructed in [2] or perhaps the estimator investigated by [6] and [1]. Let also h_n be a sequence of positive real numbers converging to 0, that is,

$$h_n \downarrow 0 \quad (1.2)$$

as $n \rightarrow \infty$. With these notations, we now define an estimator of $\lambda(s)$ as

$$\hat{\lambda}_n(s) := \frac{\hat{\tau}_n}{n} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} X([s + k\hat{\tau}_n - h_n, s + k\hat{\tau}_n + h_n] \cap [0, n]). \quad (1.3)$$

Let us now describe the idea behind the construction of the estimator $\hat{\lambda}_n(s)$. Note that, since there is only one realization of the Poisson process X available, we have to combine information about the (unknown) value of $\lambda(s)$ from different places of the window $[0, n]$. For this reason, the periodicity of λ , that is assumption (1.1), plays a crucial role and leads to the following string of (approximate) equations

$$\begin{aligned} \lambda(s) &= \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \lambda(s + k\tau) \mathbf{I}\{s + k\tau \in [0, n]\} \\ &\approx \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} \int_{[s+k\tau-h_n, s+k\tau+h_n] \cap [0, n]} \lambda(x) dx \\ &= \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} \mathbf{E}X([s + k\tau - h_n, s + k\tau + h_n] \cap [0, n]) \\ &\approx \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} X([s + k\tau - h_n, s + k\tau + h_n] \cap [0, n]) \\ &\approx \frac{\tau}{n} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} X([s + k\tau - h_n, s + k\tau + h_n] \cap [0, n]), \quad (1.4) \end{aligned}$$

where

$$N_n = \#\{k : s + k\tau \in [0, n]\}.$$

We note that, in order to make the first \approx in (1.4) works, we require the assumptions that s is a Lebesgue point of λ and (1.2) holds true. We say s is a Lebesgue point of λ , if we have

$$\lim_{h \downarrow 0} \frac{1}{2h} \int_{-h}^h |\lambda(s+x) - \lambda(s)| dx = 0 \tag{1.5}$$

(eg. see [7], p.107-108). Thus, from (1.4) we conclude that the quantity

$$\lambda_n(s) := \frac{\tau}{n} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} X([s+k\tau-h_n, s+k\tau+h_n] \cap [0, n]), \tag{1.6}$$

can be viewed as an estimator of $\lambda(s)$, provided that the period τ is known. The estimator (1.3) is obtained by replacing τ in (1.6) by $\hat{\tau}_n$.

The idea described in (1.4) and (1.6) of constructing an estimator for $\lambda(s)$ resembles that of [4] where in a similar fashion a non-parametric estimator for an intensity function which, in addition to the periodic trend, also has a polynomial trend. In [4], just like when constructing the estimator $\lambda_n(s)$ in (1.6), the period τ is supposed to be known.

Theorem 1.1. *Let the intensity function λ be periodic and locally integrable. Furthermore, let the bandwidth h_n be such that (1.2) holds true, and*

$$nh_n \rightarrow \infty \tag{1.7}$$

as $n \rightarrow \infty$. If

$$n|\hat{\tau}_n - \tau|/h_n \xrightarrow{p} 0 \tag{1.8}$$

as $n \rightarrow \infty$, then

$$\hat{\lambda}_n(s) \xrightarrow{p} \lambda(s) \tag{1.9}$$

as $n \rightarrow \infty$, provided s is a Lebesgue point of λ . In other words, $\hat{\lambda}_n(s)$ is a consistent estimator of $\lambda(s)$.

2. PROOFS OF THEOREM 1.1

Let $B_h(x)$ denotes the interval $[x-h, x+h]$. To establish Theorem 1.1, first we prove

$$\frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} X(B_{h_n}(s+k\hat{\tau}_n) \cap [0, n]) \xrightarrow{p} \lambda(s), \tag{2.1}$$

as $n \rightarrow \infty$, where $N_n = \#\{k : s+k\tau \in [0, n]\}$. By Lemma 2.1, Lemma 2.2, and Lemma 2.3, we obtain that the quantity on the l.h.s. of (2.1) is equal to $\lambda(s) + o_p(1)$, as $n \rightarrow \infty$, which of course implies (2.1). Then, to prove (1.9), it remains to check that $\hat{\lambda}_n(s)$ can be replaced by the quantity on the l.h.s. of (2.1), i.e. we must show that the difference between $\hat{\lambda}_n(s)$ and the quantity on the l.h.s. of (2.1)

converges in probability to zero, as $n \rightarrow \infty$. To show this, first we write this difference as

$$\left(\frac{\hat{\tau}_n N_n}{n} - 1 \right) \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} X(B_{h_n}(s + k\hat{\tau}_n) \cap [0, n]), \quad (2.2)$$

that is, the quantity on the l.h.s. of (2.1) multiplied by $(\hat{\tau}_n N_n n^{-1} - 1)$. Since $\lambda(s)$ is finite, by (2.1), we have that the quantity on the l.h.s. of (2.1) is $\mathcal{O}_p(1)$, as $n \rightarrow \infty$. Hence, it remains to check that

$$\left| \frac{\hat{\tau}_n N_n}{n} - 1 \right| = o_p(1), \quad (2.3)$$

as $n \rightarrow \infty$. By the triangle inequality, the quantity on the l.h.s. of (2.3) does not exceed

$$\left| \frac{\hat{\tau}_n N_n}{n} - \frac{\hat{\tau}_n}{\tau} \right| + \left| \frac{\hat{\tau}_n}{\tau} - 1 \right| \leq \frac{\hat{\tau}_n}{n} \left| N_n - \frac{n}{\tau} \right| + \frac{1}{\tau} |\hat{\tau}_n - \tau|. \quad (2.4)$$

Note that $|n/\tau - N_n| \leq 1$, and $\hat{\tau}_n = \mathcal{O}_p(1)$, as $n \rightarrow \infty$ (by (1.8)). Hence, the first term on the r.h.s. of (2.4) is $\mathcal{O}_p(n^{-1})$, as $n \rightarrow \infty$. By (1.8), we have that its second term is $o_p(n^{-1})$, as $n \rightarrow \infty$. Therefore we have (2.3). This completes the proof of Theorem 1.1.

In the following lemma we shall show that we may replace the random centre $s + k\hat{\tau}_n$ of the interval $B_{h_n}(s + k\hat{\tau}_n)$ in (2.1) by its deterministic limit $s + k\tau$.

Lemma 2.1. *Suppose λ is periodic (with period τ) and locally integrable. If, in addition, (1.2) and (1.8) are satisfied, then*

$$\begin{aligned} & \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} |\{X(B_{h_n}(s + k\hat{\tau}_n) \cap [0, n]) - X(B_{h_n}(s + k\tau) \cap [0, n])\}| \\ &= o_p(1), \end{aligned} \quad (2.5)$$

as $n \rightarrow \infty$, provided s is a Lebesgue point of λ .

Proof: First note that the difference within curly brackets on the l.h.s. of (2.5) does not exceed

$$X(B_{h_n}(s + k\hat{\tau}_n) \Delta B_{h_n}(s + k\tau) \cap [0, n]). \quad (2.6)$$

Now we notice that

$$B_{h_n - |k(\hat{\tau}_n - \tau)|}(s + k\tau) \subseteq B_{h_n}(s + k\hat{\tau}_n) \subseteq B_{h_n + |k(\hat{\tau}_n - \tau)|}(s + k\tau). \quad (2.7)$$

By (2.6) and (2.7) we have

$$\begin{aligned} & |\{X(B_{h_n}(s + k\hat{\tau}_n) \cap [0, n]) - X(B_{h_n}(s + k\tau) \cap [0, n])\}| \\ & \leq 2X(B_{h_n + |k(\hat{\tau}_n - \tau)|}(s + k\tau) \setminus B_{h_n - |k(\hat{\tau}_n - \tau)|}(s + k\tau) \cap [0, n]). \end{aligned} \quad (2.8)$$

Hence, to prove (2.5), it suffices to show that

$$\begin{aligned} & \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{h_n} X(B_{h_n+|k(\hat{\tau}_n-\tau)|(s+k\tau)} \setminus B_{h_n-|k(\hat{\tau}_n-\tau)|(s+k\tau)} \cap [0, n]) \\ &= o_p(1), \end{aligned} \tag{2.9}$$

as $n \rightarrow \infty$. To prove (2.9) we argue as follows. Let Λ_n denotes the l.h.s. of (2.9), and let also $\epsilon > 0$ be any fixed real number. Then, for any fixed $\delta > 0$, we have that

$$\begin{aligned} \mathbf{P}(|\Lambda_n| \geq \epsilon) &\leq \mathbf{P}(\{|\Lambda_n| \geq \epsilon\} \cap \{n|\hat{\tau}_n - \tau| \leq \delta h_n\}) \\ &\quad + \mathbf{P}(n|\hat{\tau}_n - \tau| > \delta h_n). \end{aligned} \tag{2.10}$$

By (1.8), the second term on the r.h.s. of (2.10) is $o(1)$, as $n \rightarrow \infty$. While the first term on the r.h.s. of (2.10), does not exceed $\mathbf{P}(|\bar{\Lambda}_n| \geq \epsilon)$, where

$$\bar{\Lambda}_n = \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{h_n} X(B_{h_n+\delta h_n}(s+k\tau) \setminus B_{h_n-\delta h_n}(s+k\tau) \cap [0, n]). \tag{2.11}$$

Next, by Markov inequality for the first moment, we have that

$$\mathbf{P}(|\bar{\Lambda}_n| \geq \epsilon) \leq \epsilon^{-1} \mathbf{E}|\bar{\Lambda}_n|,$$

and $\epsilon^{-1} \mathbf{E}|\bar{\Lambda}_n|$ can also be written as

$$\begin{aligned} & \frac{1}{\epsilon N_n} \sum_{k=-\infty}^{\infty} \frac{1}{h_n} \int_{B_{(1+\delta)h_n}(0) \setminus B_{(1-\delta)h_n}(0)} \lambda(s+k\tau+x) \mathbf{I}(s+k\tau+x \in [0, n]) dx \\ &= \frac{1}{\epsilon N_n} \frac{1}{h_n} \int_{B_{(1+\delta)h_n}(0) \setminus B_{(1-\delta)h_n}(0)} \lambda(s+x) \sum_{k=-\infty}^{\infty} \mathbf{I}(s+k\tau+x \in [0, n]) dx. \end{aligned} \tag{2.12}$$

Now we can easily see that

$$\sum_{k=-\infty}^{\infty} \mathbf{I}(s+k\tau+x \in [0, n]) \leq N_n + 1.$$

Then, the r.h.s. of (2.12) does not exceed

$$\frac{1}{\epsilon h_n} \left(\frac{1}{N_n} + 1 \right) \int_{B_{(1+\delta)h_n}(0) \setminus B_{(1-\delta)h_n}(0)} \lambda(s+x) dx. \tag{2.13}$$

We also can see that $N_n^{-1} + 1 \leq 2$. Furthermore, the quantity in (2.13) can be bounded above by

$$\begin{aligned} & \frac{2}{\epsilon h_n} \int_{B_{(1+\delta)h_n}(0)} |\lambda(s+x) - \lambda(s)| dx \\ & \quad + \frac{2}{\epsilon h_n} |B_{(1+\delta)h_n}(0) \setminus B_{(1-\delta)h_n}(0)| \lambda(s). \end{aligned} \tag{2.14}$$

Since s is a Lebesgue point of λ , the first term of (2.14) converges to zero as $n \rightarrow \infty$. While the second term of (2.14) does not exceed

$8\epsilon^{-1}\delta\lambda(s)$. By taking $\delta = \delta_n \downarrow 0$ as $n \rightarrow \infty$, we also have that this term converges to zero as $n \rightarrow \infty$. Then we get that $\mathbf{P}(|\Lambda_n| \geq \epsilon) \rightarrow 0$ as $n \rightarrow \infty$, which is equivalent to (2.9). This completes the proof of Lemma 2.1.

To complete our proof of Theorem 1.1 we also need the following lemma.

Lemma 2.2. *Suppose λ is periodic (with period τ) and locally integrable. If, in addition, (1.2) and (1.7) are satisfied, then*

$$\begin{aligned} & \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} |X(B_{h_n}(s+k\tau) \cap [0, n]) - \mathbf{E}X(B_{h_n}(s+k\tau) \cap [0, n])| \\ & = o_p(1), \end{aligned} \quad (2.15)$$

as $n \rightarrow \infty$, provided s is a Lebesgue point of λ .

Proof: First note that, for large n , the random variables

$$X(B_{h_n}(s+k\tau) \cap [0, n]),$$

for all $k \in \mathbf{Z}$, are independent. Then, by Chebyshev's inequality, to prove (2.15) it suffices to check that

$$\left(\frac{1}{2N_n h_n}\right)^2 \sum_{k=-\infty}^{\infty} \text{Var}\{X(B_{h_n}(s+k\tau) \cap [0, n])\} = o(1), \quad (2.16)$$

as $n \rightarrow \infty$. Since X is a Poisson random variable, $\text{Var}(X) = \mathbf{E}X$, and for each k , we can write

$$\begin{aligned} & \mathbf{E}X(B_{h_n}(s+k\tau) \cap [0, n]) \\ & = \int_{B_{h_n}(0)} \lambda(s+k\tau+x) \mathbf{I}(s+k\tau+x \in [0, n]) dx. \end{aligned} \quad (2.17)$$

Because λ is periodic (with period τ), we have that $\lambda(s+k\tau+x) = \lambda(s+x)$, and we also have that $\sum_{k=-\infty}^{\infty} \mathbf{I}(s+k\tau+x \in [0, n]) \leq N_n + 1$. Then, to prove (2.16), it suffices to show

$$\frac{1}{2} \left(\frac{N_n+1}{N_n}\right) \left(\frac{1}{N_n h_n}\right) \left(\frac{1}{2h_n} \int_{B_{h_n}(0)} \lambda(s+x) dx\right) = o(1), \quad (2.18)$$

as $n \rightarrow \infty$. Because s is a Lebesgue point of λ , we have

$$(2h_n)^{-1} \int_{B_{h_n}(0)} \lambda(s+x) dx = \lambda(s) + o(1),$$

as $n \rightarrow \infty$, which is finite. Because $N_n h_n \rightarrow \infty$ as $n \rightarrow \infty$, (by (1.7)), then we get (2.18). This completes the proof of Lemma 2.2.

It remains to evaluate a non-random sum.

Lemma 2.3. *Suppose λ is periodic (with period τ) and locally integrable. If, in addition, (1.2) is satisfied, then*

$$\frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} \mathbf{E}X(B_{h_n}(s+k\tau) \cap [0, n]) = \lambda(s) + o(1), \quad (2.19)$$

as $n \rightarrow \infty$, provided s is a Lebesgue point of λ .

Proof: Using the fact that X is Poisson, the l.h.s. of (2.19) can be written as

$$\begin{aligned} & \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} \int_{-h_n}^{h_n} \lambda(s+k\tau+x) \mathbf{I}(s+k\tau+x \in [0, n]) dx \\ &= \frac{1}{2N_n h_n} \int_{-h_n}^{h_n} \lambda(s+x) \sum_{k=-\infty}^{\infty} \mathbf{I}(s+k\tau+x \in [0, n]) dx. \end{aligned} \quad (2.20)$$

Now note that

$$(N_n - 1) \leq \sum_{k=-\infty}^{\infty} \mathbf{I}(s+k\tau+x \in [0, n]) \leq (N_n + 1),$$

which implies $N_n^{-1} \sum_{k=-\infty}^{\infty} \mathbf{I}(s+k\tau+x \in [0, n])$ can be written as $(1 + \mathcal{O}(n^{-1}))$, as $n \rightarrow \infty$, uniformly in x . Then, the quantity on the r.h.s. of (2.20) can be written as

$$\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right) \frac{1}{2h_n} \int_{-h_n}^{h_n} \lambda(s+x) dx. \quad (2.21)$$

By (1.2) together with the assumption that s is a Lebesgue point of λ , we have that $(2h_n)^{-1} \int_{-h_n}^{h_n} \lambda(s+x) dx = \lambda(s) + o(1)$, as $n \rightarrow \infty$. Then we get this lemma. This completes the proof of Lemma 2.3.

REFERENCES

- [1] M. Bebbington and R. Zitikis (2004). A robust heuristic estimator for the period of a Poisson intensity function. *Methodology and Computing in Applied Probability*, **6**, 441-462.
- [2] R. Helmers and I W. Mangku (2003), On estimating the period of a cyclic Poisson process. *Mathematical Statistics and Applications: Festschrift in honor of Constance van Eeden*. (Editors: Marc Moore, Sorana Froda and Christian Leger), IMS Lecture Notes Series - Monograph Series, Volume 42.
- [3] R. Helmers, I W. Mangku, and R. Zitikis (2003), Consistent estimation of the intensity function of a cyclic Poisson process. *J. Multivariate Anal.* **84**, 19-39.
- [4] R. Helmers and R. Zitikis (1999). On estimation of Poisson intensity functions, *Ann. Inst. Stat. Math.*, **51**, 2, 265-280.
- [5] Mangku, I W. (2001). *Estimating the Intensity of a Cyclic Poisson Process*, Ph.D Thesis, University of Amsterdam, The Netherlands.
- [6] D. Vere-Jones (1982). On the estimation of frequency in point-process data. *J. Appl. Probab.* **19A**, 383-394.
- [7] R. L. Wheeden and A. Zygmund (1977), *Measure and Integral: An Introduction to Real Analysis*. Marcel Dekker, Inc., New York.