CONSISTENCY OF KERNEL-TYPE ESTIMATORS FOR THE FIRST AND SECOND DERIVATIVES OF A PERIODIC POISSON INTENSITY FUNCTION

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Abstract. We construct and investigate consistent kernel-type estimators for the first and second derivatives of a periodic Poisson intensity function when the period is known. We do not assume any particular parametric form for the intensity function. Moreover, we consider the situation when only a single realization of the Poisson process is available, and only observed in a bounded interval. We prove that the proposed estimators are consistent when the length of the interval goes to infinity. We also prove that the mean-squared error of the estimators converge to zero when the length of the interval goes to infinity.

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1. Introduction

We consider kernel type estimations for the first and second derivatives of the intensity function of a periodic Poisson process. Let N be a Poisson process on $[0, \infty)$ with (unknown) locally integrable intensity function λ. We assume that λ is a periodic function with (known) period τ. We do not assume any parametric form of λ, except that it is periodic. That is, for each point $s \in [0, \infty)$ and all $k \in \mathbf{Z}$, with $\mathbf{Z}$ denotes the set of integers, we have

$$\lambda(s + k\tau) = \lambda(s). \tag{1.1}$$

Suppose that, for some $\omega \in \Omega$, a single realization $N(\omega)$ of the Poisson process N defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with intensity function λ is observed, though only within a bounded interval $[0, n]$. Our goal in this paper is to study consistency of estimators for the first and second derivatives of the intensity function λ at a given point

$s \in [0, \infty)$ using only a single realization $N(\omega)$ of the Poisson process N observed in interval $[0, n]$. A special case study using uniform kernel estimators can be found in [4].

Throughout this paper, we assume that s is a Lebesgue point of λ, that is we have

$$\lim_{h \downarrow 0} \frac{1}{2h} \int_{-h}^{h} |\lambda(s+x) - \lambda(s)| dx = 0 \tag{1.2}$$

(eg. see [7], p.107-108).

Since λ is a periodic function with period τ, the problem of estimating λ, λ' (the first derivative of λ) and λ'' (the second derivative of λ) at a given point $s \in [0, \infty)$ can be reduced into a problem of estimating λ, λ' and λ'' at a given point $s \in [0, \tau)$. Hence, for the rest of this paper, we assume that $s \in [0, \tau)$.

2. The estimators and some results

To define estomators of λ' and λ'' we need an estimator of λ. Therefore, before defining estomators of λ' and λ'', we first review the construction of a kernel-type estimator of λ at a given point s, as given in Helmers *et al.* [2], as follows. Let $K : \mathbf{R} \to \mathbf{R}$ be a real valued function, called *kernel*, which satisfies the following conditions: (K1) K is a probability density function, (K2) K is bounded, and (K3) K has (closed) support $[-1, 1]$. Let also h_n be a sequence of positive real numbers converging to 0, that is,

$$h_n \downarrow 0, \tag{2.1}$$

as $n \to \infty$. Now, we may define the estimator of λ at a given point $s \in [0, \tau)$ as follows

$$\hat{\lambda}_{n,K}(s) := \frac{\tau}{n} \sum_{k=0}^{\infty} \frac{1}{h_n} \int_0^n K\left(\frac{x - (s + k\tau)}{h_n}\right) N(dx). \tag{2.2}$$

This estimator is a special case of a more general kernel-type estimator of the intensity of a periodic Poisson process, which includes the case when the period τ has to be estimated (see Helmers *et al.* ([2], [3])).

By having the estimator of λ at a given point $s \in [0, \tau)$, following the idea in Helmers and Mangku [1], we may define an estimator of λ' at a given point $s \in [0, \tau)$ as follows

$$\hat{\lambda}'_{n,K}(s) := \frac{\hat{\lambda}_{n,K}(s + h_n) - \hat{\lambda}_{n,K}(s - h_n)}{2h_n}. \tag{2.3}$$

Construction of this estimator is using the fact that, for small h we have

$$\lambda'(s) \approx \frac{\lambda(s+h) - \lambda(s-h)}{2h}.$$

Consistency of $\hat{\lambda}'_{n,K}(s)$ is given in the following theorem.

Theorem 2.1. (Consistency of $\hat{\lambda}'_{n,K}(s)$)

Suppose that the intensity function λ is periodic and locally integrable, and has finite first derivative λ' at s. If the kernel K is symmetric and satisfies conditions $(K1), (K2), (K3)$, and h_n satisfies assumptions (2.1) and $nh_n^3 \to \infty$, then

$$\hat{\lambda}'_{n,K}(s) \xrightarrow{p} \lambda'(s), \tag{2.4}$$

as $n \to \infty$. In other words, $\hat{\lambda}'_{n,K}(s)$ is a consistent estimator of $\lambda'(s)$. In addition, the mean-squared error (MSE) of $\hat{\lambda}'_{n,K}(s)$ converges to 0, as $n \to \infty$.

Next we consider estimation of the second derivative λ'' of λ at a given point $s \in [0, \tau)$. Following the idea in Helmers and Mangku [1], we may define an estimator of λ'' at a given point $s \in [0, \tau)$ as follows

$$\hat{\lambda}''_{n,K}(s) := \frac{\hat{\lambda}_{n,K}(s + 2h_n) + \hat{\lambda}_{n,K}(s - 2h_n) - 2\hat{\lambda}_{n,K}(s)}{4h_n^2}. \tag{2.5}$$

Construction of this estimator is using the fact that, for small h we have

$$\lambda''(s) \approx \frac{\lambda'(s + h) - \lambda'(s - h)}{2h} \approx \frac{\lambda(s + 2h) + \lambda(s - 2h) - 2\lambda(s)}{4h^2}.$$

Consistency of $\hat{\lambda}''_{n,K}(s)$ is given in the following theorem.

Theorem 2.2. (Consistency of $\hat{\lambda}''_{n,K}(s)$)

Suppose that the intensity function λ is periodic and locally integrable, and has finite second derivative λ'' at s. If the kernel K is symmetric and satisfies conditions $(K1), (K2), (K3)$, and h_n satisfies assumptions (2.1) and $nh_n^5 \to \infty$, then

$$\hat{\lambda}''_{n,K}(s) \xrightarrow{p} \lambda''(s), \tag{2.6}$$

as $n \to \infty$. In other words, $\hat{\lambda}''_{n,K}(s)$ is a consistent estimator of $\lambda''(s)$. In addition, the MSE of $\hat{\lambda}''_{n,K}(s)$ converges to 0, as $n \to \infty$.

3. Some Technical Lemmas

To prove Theorems 2.1 and 2.2 we need the following two lemmas. The first lemma is about asymptotic approximations to $\mathbf{E}\hat{\lambda}_{n,K}(s)$ in two cases, namely (i) when λ has finite first derivative at s, (ii) when λ has finite second derivative at s. The second lemma is about asymptotic approximation to the variance of $\hat{\lambda}_{n,K}(s)$. We will use the first lemma to prove that the bias of $\hat{\lambda}'_{n,K}(s)$ and $\hat{\lambda}''_{n,K}(s)$ converge to zero as $n \to \infty$.

The second lemma will be used to prove that the variances of $\hat{\lambda}'_{n,K}(s)$ and $\hat{\lambda}''_{n,K}(s)$ converge to zero as $n \to \infty$.

Lemma 3.1. (Asymptotic approximations to the bias of $\hat{\lambda}_{n,K}(s)$)

Suppose that the intensity function λ is periodic and locally integrable, the kernel K is symmetric and satisfies conditions $(K1), (K2), (K3)$, and h_n satisfies assumptions (2.1).

(i) *If $nh_n \to \infty$ and λ has finite first derivative at s then*

$$\mathbf{E}\hat{\lambda}_{n,K}(s) = \lambda(s) + o(h_n), \tag{3.1}$$

as $n \to \infty$.

(ii) *If $nh_n^2 \to \infty$ and λ has finite second derivative at s then*

$$\mathbf{E}\hat{\lambda}_{n,K}(s) = \lambda(s) + \frac{1}{2}\lambda''(s)h_n^2 \int_{-1}^{1} x^2 K(x)dx + o(h_n^2), \tag{3.2}$$

as $n \to \infty$.

Proof: Here we only give the proof of part (i) of this lemma (see also [6]). Proof of part (ii) of this lemma can be found in [5]. To prove (3.1), first note that

$$\begin{aligned}
\mathbf{E}\hat{\lambda}_{n,K}(s) &= \frac{\tau}{n}\sum_{k=0}^{\infty}\frac{1}{h_n}\int_0^n K\left(\frac{x-(s+k\tau)}{h_n}\right)\mathbf{E}N(dx) \\
&= \frac{\tau}{n}\sum_{k=0}^{\infty}\frac{1}{h_n}\int_0^n K\left(\frac{x-(s+k\tau)}{h_n}\right)\lambda(x)dx \\
&= \frac{\tau}{n}\sum_{k=0}^{\infty}\frac{1}{h_n}\int_{\mathbf{R}} K\left(\frac{x-(s+k\tau)}{h_n}\right)\lambda(x)\mathbf{I}(x \in [0,n])dx.
\end{aligned} \tag{3.3}$$

By a change of variable and using (1.1), we can write the r.h.s. of (3.3) as

$$\begin{aligned}
&\frac{\tau}{n}\sum_{k=0}^{\infty}\frac{1}{h_n}\int_{\mathbf{R}} K\left(\frac{x}{h_n}\right)\lambda(x+s+k\tau)\mathbf{I}(x+s+k\tau \in [0,n])dx \\
=\ &\frac{\tau}{n}\sum_{k=0}^{\infty}\frac{1}{h_n}\int_{\mathbf{R}} K\left(\frac{x}{h_n}\right)\lambda(x+s)\mathbf{I}(x+s+k\tau \in [0,n])dx \\
=\ &\frac{\tau}{nh_n}\int_{\mathbf{R}} K\left(\frac{x}{h_n}\right)\lambda(x+s)\sum_{k=0}^{\infty}\mathbf{I}(x+s+k\tau \in [0,n])dx.
\end{aligned} \tag{3.4}$$

Now note that

$$\sum_{k=0}^{\infty}\mathbf{I}(x+s+k\tau \in [0,n]) \in \left[\frac{n}{\tau}-1, \frac{n}{\tau}-1\right]. \tag{3.5}$$

Then, the r.h.s. of (3.4) can be written as

$$\begin{aligned}
&\frac{\tau}{n\ h_n}\int_{\mathbf{R}} K\left(\frac{x}{h_n}\right)\lambda(x+s)\left(\frac{n}{\tau}+\mathcal{O}(1)\right)dx \\
=\ &\frac{1}{h_n}\int_{\mathbf{R}} K\left(\frac{x}{h_n}\right)\lambda(x+s)dx+\mathcal{O}\left(\frac{1}{n}\right) \\
=\ &\int_{-1}^{1} K(x)\lambda(s+xh_n)dx+\mathcal{O}\left(\frac{1}{n}\right), \qquad (3.6)
\end{aligned}$$

as $n \to \infty$. By the Young's form of Taylor's theorem, we have

$$\lambda(s+xh_n)=\lambda(s)+\frac{\lambda'(s)}{1!}xh_n+o(h_n), \qquad (3.7)$$

jika $n \to \infty$. Substituting (3.7) into the r.h.s. of (3.6), we obtain

$$\begin{aligned}
\mathbf{E}\hat{\lambda}_{n,K}(s) &= \int_{-1}^{1} K(x)\left(\lambda(s)+\frac{\lambda'(s)}{1!}xh_n\right)dx+o(h_n)+\mathcal{O}\left(\frac{1}{n}\right) \\
&= \lambda(s)\int_{-1}^{1}K(x)dx+\lambda'(s)h_n\int_{-1}^{1}xK(x)dx+o(h_n) \\
&\quad +\mathcal{O}\left(\frac{1}{n}\right), \qquad (3.8)
\end{aligned}$$

as $n \to \infty$. By assumption (K1) and (K3) we have $\int_{-1}^{1} K(x)dx = 1$. Since the kernel K is symmetric, an easy calculation shows that the second term on the r.h.s. of (3.8) is equal to zero. By the assumption $nh_n \to \infty$, we have the last term on the r.h.s. of (3.8) is of order $o(h_n)$, as $n \to \infty$. Hence we obtain (3.1). This completes the proof of part (i) of Lemma 3.1.

Lemma 3.2. (Asymptotic approximation to the variance of $\hat{\lambda}_{n,K}(s)$)

Suppose that the intensity function λ is periodic and locally integrable. If the kernel K satisfies conditions $(K1),(K2),(K3)$, and h_n satisfies assumptions (2.1), then

$$Var\left(\hat{\lambda}_{n,K}(s)\right)=\frac{\tau\lambda(s)}{nh_n}\int_{-1}^{1}K^2(x)dx+o\left(\frac{1}{nh_n}\right) \qquad (3.9)$$

as $n \to \infty$, provided s is a Lebesgue point of λ.

Proof: We refer to [5] for the proof of this lemma.

4. Proof of Theorem 2.1

To prove Theorem 2.1 it suffices to check the following two lemmas.

Lemma 4.1. (Asymptotic unbiasedness of $\hat{\lambda}'_{n,K}(s)$)

Suppose that the intensity function λ is periodic, locally integrable and has finite first derivative at s. If the kernel K is symmetric and satisfies *conditions* $(K1), (K2), (K3)$*, the bandwidth* h_n *satisfies assumptions (2.1) and* $nh_n \to \infty$*, then*

$$\mathbf{E}\hat{\lambda}'_{n,K}(s) \;\to\; \lambda'(s), \tag{4.1}$$

as $n \to \infty$*. In other words,* $\hat{\lambda}'_{n,K}(s)$ *is asymptotically unbiased estimator of* $\lambda'(s)$*.*

Proof: By (2.3), $\mathbf{E}\hat{\lambda}'_{n,K}(s)$ can be computed as follows

$$\mathbf{E}\hat{\lambda}'_{n,K}(s) = \frac{1}{2h_n}\Big(\mathbf{E}\hat{\lambda}_{n,K}(s+h_n) - \mathbf{E}\hat{\lambda}_{n,K}(s-h_n)\Big). \tag{4.2}$$

By (3.1) and Taylor expansion we have

$$\mathbf{E}\hat{\lambda}_{n,K}(s+h_n) = \lambda(s+h_n) + o(h_n) = \lambda(s) + \frac{\lambda'(s)}{1!}h_n + o(h_n) \tag{4.3}$$

and

$$\mathbf{E}\hat{\lambda}_{n,K}(s-h_n) = \lambda(s-h_n) + o(h_n) = \lambda(s) - \frac{\lambda'(s)}{1!}h_n + o(h_n) \tag{4.4}$$

as $n \to \infty$. Substituting (4.3) and (4.4) into the r.h.s. of (4.2), then we obtain

$$\mathbf{E}\hat{\lambda}'_{n,K}(s) = \lambda'(s) + o(1)$$

as $n \to \infty$, which is equivalent to (4.1). This completes the proof of Lemma 4.1.

Lemma 4.2. (Convergency of the variance of $\hat{\lambda}'_{n,K}(s)$)

Suppose that the intensity function λ *is periodic and locally integrable. If the kernel* K *satisfies conditions* $(K1), (K2), (K3)$*, and* h_n *satisfies assumptions (2.1) and* $nh_n^3 \to \infty$*, then*

$$Var\left(\hat{\lambda}'_{n,K}(s)\right) \;\to\; 0, \tag{4.5}$$

as $n \to \infty$*, provided* s *is a Lebesgue point of* λ*.*

Proof: By (2.3), $Var(\hat{\lambda}'_{n,K}(s))$ can be computed as follows

$$\begin{aligned} Var\left(\hat{\lambda}'_{n,K}(s)\right) &= \frac{1}{4h_n^2}\Big(Var(\hat{\lambda}_{n,K}(s+h_n)) + Var(\hat{\lambda}_{n,K}(s-h_n)) \\ &\quad -2Cov(\hat{\lambda}_{n,K}(s+h_n), \hat{\lambda}_{n,K}(s-h_n))\Big). \end{aligned} \tag{4.6}$$

By (2.1), for sufficiently large n, we have that for each integer k, the interval $[s+k\tau, s+k\tau+2h_n]$ and $[s+k\tau-2h_n, s+k\tau]$ are disjoint. This means that $\hat{\lambda}_{n,K}(s+h_n)$ and $\hat{\lambda}_{n,K}(s-h_n)$ are independent, which implies $Cov(\hat{\lambda}_{n,K}(s+h_n), \hat{\lambda}_{n,K}(s-h_n)) = 0$. Then (4.6) reduces to

$$Var\left(\hat{\lambda}'_{n,K}(s)\right) = \frac{1}{4h_n^2}\Big(Var(\hat{\lambda}_{n,K}(s+h_n)) + Var(\hat{\lambda}_{n,K}(s-h_n))\Big) \tag{4.7}$$

By (3.9) we obtain that

$$Var\left(\hat{\lambda}'_{n,K}(s)\right) = \frac{1}{4h_n^2}\mathcal{O}\left(\frac{1}{nh_n}\right) = \mathcal{O}\left(\frac{1}{nh_n^3}\right),$$

as $n \to \infty$. By the assumption $nh_n^3 \to \infty$, we obtain (4.5). This completes the proof of Lemma 4.2.

5. Proof of Theorem 2.2

To prove Theorem 2.2 it suffices to check the following two lemmas.

Lemma 5.1. (Asymptotic unbiasedness of $\hat{\lambda}''_{n,K}(s)$)

Suppose that the intensity function λ is periodic, locally integrable and has finite second derivative at s. If the kernel K is symmetric and satisfies conditions $(K1), (K2), (K3)$, the bandwidth h_n satisfies assumptions (2.1) and $nh_n^2 \to \infty$, then

$$\mathbf{E}\hat{\lambda}''_{n,K}(s) \;\to\; \lambda''(s), \tag{5.1}$$

as $n \to \infty$. In other words, $\hat{\lambda}''_{n,K}(s)$ is asymptotically unbiased estimator of $\lambda''(s)$.

Proof: By (2.5), $\mathbf{E}\hat{\lambda}''_{n,K}(s)$ can be computed as follows

$$\mathbf{E}\hat{\lambda}''_{n,K}(s) = \frac{1}{4h_n^2}\left(\mathbf{E}\hat{\lambda}_{n,K}(s+2h_n) + \mathbf{E}\hat{\lambda}_{n,K}(s-2h_n) - 2\mathbf{E}\hat{\lambda}_{n,K}(s)\right) \tag{5.2}$$

By (3.2) we have

$$\begin{aligned}\mathbf{E}\hat{\lambda}_{n,K}(s+2h_n) &= \lambda(s+2h_n) + \frac{1}{2}\lambda''(s+2h_n)h_n^2\int_{-1}^{1} x^2K(x)dx \\ &\quad + o(h_n^2), \end{aligned} \tag{5.3}$$

and

$$\begin{aligned}\mathbf{E}\hat{\lambda}_{n,K}(s-2h_n) &= \lambda(s-2h_n) + \frac{1}{2}\lambda''(s-2h_n)h_n^2\int_{-1}^{1} x^2K(x)dx \\ &\quad + o(h_n^2), \end{aligned} \tag{5.4}$$

as $n \to \infty$. By Taylor expansion we obtain

$$\lambda(s+2h_n) = \lambda(s) + \frac{\lambda'(s)}{1!}2h_n + \frac{\lambda''(s)}{2!}4h_n^2 + o(h_n^2), \tag{5.5}$$

$$\lambda(s-2h_n) = \lambda(s) - \frac{\lambda'(s)}{1!}2h_n + \frac{\lambda''(s)}{2!}4h_n^2 + o(h_n^2), \tag{5.6}$$

$$\lambda''(s+2h_n) = \lambda''(s) + o(1), \tag{5.7}$$

$$\lambda''(s-2h_n) = \lambda''(s) + o(1), \tag{5.8}$$

as $n \to \infty$. Substituting (5.5) and (5.7) into the r.h.s. of (5.3), we obtain

$$\begin{aligned}\mathbf{E}\hat{\lambda}_{n,K}(s+2h_n) &= \lambda(s) + \frac{\lambda'(s)}{1!}2h_n + \frac{\lambda''(s)}{2!}4h_n^2 \\ &\quad + \frac{1}{2}\lambda''(s)h_n^2 \int_{-1}^{1} x^2 K(x)dx + o(h_n^2), \quad (5.9)\end{aligned}$$

as $n \to \infty$. Substituting (5.6) and (5.8) into the r.h.s. of (5.4), we obtain

$$\begin{aligned}\mathbf{E}\hat{\lambda}_{n,K}(s-2h_n) &= \lambda(s) - \frac{\lambda'(s)}{1!}2h_n + \frac{\lambda''(s)}{2!}4h_n^2 \\ &\quad + \frac{1}{2}\lambda''(s)h_n^2 \int_{-1}^{1} x^2 K(x)dx + o(h_n^2), \quad (5.10)\end{aligned}$$

as $n \to \infty$. Finally, by substituting (3.2), (5.9) and (5.10) into the r.h.s. of (5.2), we obtain

$$\mathbf{E}\hat{\lambda}''_{n,K}(s) = \lambda''(s) + o(1)$$

as $n \to \infty$, which is equivalent to (5.1). This completes the proof of Lemma 5.1.

Lemma 5.2. (Convergency of the variance of $\hat{\lambda}''_{n,K}(s)$)

Suppose that the intensity function λ is periodic and locally integrable. If the kernel K satisfies conditions $(K1), (K2), (K3)$, and h_n satisfies assumptions (2.1) and $nh_n^5 \to \infty$, then

$$Var\left(\hat{\lambda}''_{n,K}(s)\right) \to 0, \quad (5.11)$$

as $n \to \infty$, provided s is a Lebesgue point of λ.

Proof: By a simple argument, see proof of Lemma 4.2, for sufficiently large n we have that $\hat{\lambda}_{n,K}(s+2h_n)$, $\hat{\lambda}_{n,K}(s-2h_n)$ and $\hat{\lambda}_{n,K}(s)$ are independent. Then, by (2.5), $Var(\hat{\lambda}''_{n,K}(s))$ can be computed as follows

$$\begin{aligned}Var\left(\hat{\lambda}''_{n,K}(s)\right) &= \frac{1}{16h_n^4}\Big(Var(\hat{\lambda}_{n,K}(s+2h_n)) + Var(\hat{\lambda}_{n,K}(s-2h_n)) \\ &\quad + 4Var(\hat{\lambda}_{n,K}(s))\Big). \quad (5.12)\end{aligned}$$

By (3.9) we obtain that

$$Var\left(\hat{\lambda}''_{n,K}(s)\right) = \frac{1}{16h_n^4}\mathcal{O}\left(\frac{1}{nh_n}\right) = \mathcal{O}\left(\frac{1}{nh_n^5}\right),$$

as $n \to \infty$. By the assumption $nh_n^5 \to \infty$, we obtain (5.11). This completes the proof of Lemma 5.2.

References

[1] R. Helmers and I W. Mangku (2007), Estimating the intensity of a cyclic Poisson process in the presence of linear trend. Accepted by *Annals Institute of Statistical Mathematics.*

[2] R. Helmers, I W. Mangku, and R. Zitikis (2003), Consistent estimation of the intensity function of a cyclic Poisson process. *J. Multivariate Anal.* **84**, 19-39.

[3] R. Helmers, I W. Mangku, and R. Zitikis (2005), Statistical properties of a kernel-type estimator of the intensity function of a cyclic Poisson process. *J. Multivariate Anal.*, **92**, 1-23.

[4] Herniwati (2007). *Kekonsistenan Penduga Turunan Pertama dan Turunan Kedua dari Fungsi Intensitas suatu Proses Poisson Periodik.* Department of Mathematics IPB. Skripsi. Bogor.

[5] Mangku, I W. (2006). Asymptotic normality of a kernel-type estimator for the intensity of a periodic Poisson process. *Journal of Mathematics and Its Applications,* **5,** No. 2, 13-22.

[6] Syamsuri (2007). *Pendugaan Turunan Pertama dan Turunan Kedua dari Fungsi Intensitas suatu Proses Poisson Periodik.* Department of Mathematics IPB. Tesis. Bogor.

[7] R. L. Wheeden and A. Zygmund (1977), *Measure and Integral: An Introduction to Real Analysis.* Marcel Dekker, Inc., New York.