OPTIMAL ENERGY REGULATION PERFORMANCE OF DELAY-TIME SYSTEMS

TONI BAKHTIAR

Departemen of Mathematics,
Faculty of Mathematics and Natural Sciences,
Bogor Agricultural University
Darmaga, Bogor 16680, Indonesia

ABSTRACT. This paper studies the regulation performance limitation of delay-time systems. The performance is measured by the energy of the control input with respect to an impulse disturbance function. We first provide the analytical closed-form expression of the optimal performance for minimum phase case by reviewing the existing result. We then extend the problem to non-minimum phase case by exploiting the results of linear time-invariant discrete-time and delta domain cases.
Keywords: Performance limitation, regulation problem, delay-time system.

1. INTRODUCTION

Problems concerning the fundamental performance limitation and trade-off in feedback control systems have been intensively studied for decades, beginning with the work of Bode on logarithmic sensitivity integrals [5]. There are two main research directions in the area. First direction lies in the extensions of the Bode's integral theorem to assess design constraints and performance limitations via logarithmic type integrals. Second direction focuses on the formulations of optimal control problems to quantify and characterize the fundamental performance limits in terms of plant properties.

This kind of researches relates to the plant/controller design integration, where the main attention is not to design a robust or optimal controller but to design a plant which is easily controllable in practice.

The $\mathcal{H}_2$ energy regulation problem, whose objective is to minimize the energy of the control input, has attained much attention in the recent years. Its performance limitation achievable by linear time-invariant (LTI) feedback control has been intensively investigated, which led to some complete results for single-input single-output (SISO) and single-input multiple-output (SIMO) [2, 3, 4, 6, 10, 12]. These results then has been extended to multiple-input multiple-output (MIMO) case [8]. The only paper which regards the existence of a time-delay in the loop is [7].

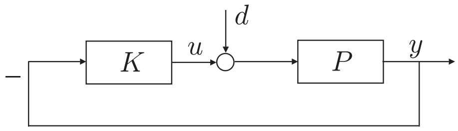

FIGURE 1. Unity feedback control system

The present paper reviews the results provided in [7] and then extends the problem to a non-minimum phase case by exploiting the discrete-time and delta domain LTI results given in [3].

After briefly stating the problem formulation in Section 2, we provide the main result for minimum phase case in Section 3. The extended result to non-minimum phase case is presented in Section 4. Some concluding statements are in Section 5.

2. Problem Formulation

2.1. **Feedback Control Setup.** We consider the unity feedback control system depicted in Fig. 1, where P is the SISO plant to be controlled with delay in the input port and K is the stabilizing controller. The plant P can be written as

$$P(s) = P_0(s)\mathrm{e}^{-s\tau}, \tag{1}$$

where $\tau \geq 0$ is a fixed time-delay, and $P_0(s)$ is a rational and strictly proper transfer function. The signals $u \in \mathbb{R}$, $d \in \mathbb{R}$, and $y \in \mathbb{R}$ are the control input, the impulse disturbance input, and the measurable output, respectively.

The problems to be investigated in this paper is the standard $\mathcal{H}_2$ optimal energy regulation problems, in which we minimize the performance index

$$E := \int_0^\infty |u(t)|^2 \,\mathrm{d}t \tag{2}$$

with respect to an impulse disturbance input d. The problems without delay have been studied in [3, 10].

From Parseval's identity [11] we can deduce that the best achievable regulation performance by all stabilizing controller K in set $\mathcal{K}$ is given by

$$E^* = \inf_{K \in \mathcal{K}} \left\| \frac{K(s)P(s)}{1 + K(s)P(s)} \right\|_2^2 . \tag{3}$$

2.2. **Coprime Factorization.** The key instrument to derive the analytical closed-form expression of the optimal performance is a coprime factorization of a plant with a time delay.

Suppose that the state space realization of $P_0(s)$ is given by

$$P_0(s) = \left(\begin{array}{c|c} A & B \\ \hline C & 0 \end{array} \right),$$

or in other words, the state space representation of P_0 is given by

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k), \end{aligned} \tag{4}$$

where x, u, y are state, input, and output variables, respectively, and A, B, C are their corresponding matrices.

and let L be any constant matrix such that $A - LC$ is Hurwitz (stable). Introduce the transfer functions

$$N(s) := \mathrm{e}^{-s\tau} N_0(s), \tag{5}$$

$$M(s) := \frac{1}{1 + C(sI - A)^{-1}L}, \tag{6}$$

where $N_0(s) = C(sI - A + LC)^{-1}B$. It is obvious that $N(s), M(s) \in \mathbb{R}\mathcal{H}_\infty$ since, by definition, their poles are the eigenvalues of $A - LC$, which is Hurwitz.

By using the so-called Sherman-Morrison-Woodbury formula [9] on the matrix $(sI - A + LC)^{-1}$ we can write

$$N_0(s) = \frac{C(sI - A)^{-1}B}{1 + C(sI - A)^{-1}L}.$$

Hence,

$$\frac{N(s)}{M(s)} = \mathrm{e}^{-s\tau} C(sI - A)^{-1}B = P(s),$$

which shows that $N(s)$ and $M(s)$ are the coprime factors of $P(s)$. Let R be a stabilising state feedback matrix such that $(A - BR)$ has the same eigenvalues as $(A - LC)$. Then, there exist transfer functions $X(s), Y(s) \in \mathbb{R}\mathcal{H}_\infty$ defined by

$$X(s) := R\mathrm{e}^{A\tau}(sI - A + LC)^{-1}L, \tag{7}$$

$$\begin{aligned} Y(s) := {} & 1 + \mathrm{e}^{A\tau}(sI - A + LC)^{-1}B\mathrm{e}^{-s\tau} + \\ & R(I - \mathrm{e}^{-(sI-A)\tau})(sI - A)^{-1}B, \end{aligned} \tag{8}$$

such that the Bezout identity

$$N(s)X(s) + M(s)Y(s) = 1 \tag{9}$$

is satisfied. See [7] for the complete overview.

From the above coprime factorization, the set of all stabilizing controllers $\mathcal{K}$ is characterized as

$$\mathcal{K} := \left\{ K(s) \mid K(s) := \frac{X(s) + M(s)Q(s)}{Y(s) - N(s)Q(s)} \right\}, \tag{10}$$

where $Q \in \mathbb{R}\mathcal{H}_\infty$ is a free parameter. Consequently, the optimal performance (3) can be further written as

$$E^* = \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \|1 - MY + MNQ\|_2^2. \tag{11}$$

3. Minimum Phase Case

In this part we assume that $P_0(s)$ has no zeros in the closed right half of the complex plane $\mathbb{C}_+$. To facilitate our derivation, we introduce the minimal state space realization of P_0 as follows

$$P_0(s) = \left(\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right) = \left(\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & 0 \end{array}\right),$$

where the spectrum of A_1 consists of all the unstable poles of P_0. We introduce the state feedback matrix

$$F = B^{\mathrm{T}}\mathcal{P}, \tag{12}$$

where

$$\mathcal{P} = \begin{pmatrix} \mathcal{P}_1 & 0 \\ 0 & 0 \end{pmatrix}$$

and $\mathcal{P}_1$ is the unique symmetric and positive definite solution of the Riccati equation

$$\mathcal{P}_1 A_1 + A_1^{\mathrm{T}}\mathcal{P}_1 = \mathcal{P}_1 B_1 B_1^{\mathrm{T}}\mathcal{P}_1. \tag{13}$$

Theorem 1. *Suppose that the plant P given in (1) is minimum phase and has unstable poles $p_k \in \mathbb{C}_+$ $(k = 1, \ldots, n_p)$. Then,*

$$E^* = 2\sum_{k=1}^{n_p} p_k + \int_0^\tau F\mathrm{e}^{At}BB^{\mathrm{T}}\mathrm{e}^{A^{\mathrm{T}}t}F^{\mathrm{T}}\,\mathrm{d}t. \tag{14}$$

Proof. We may write (11) as

$$\begin{aligned} E^* &= \inf_{Q\in\mathbb{R}\mathcal{H}_\infty} \|M^{-1} - Y + NQ\|_2^2 \\ &= \|M^{-1} - 1\|_2^2 + \inf_{Q\in\mathbb{R}\mathcal{H}_\infty} \|1 - Y + NQ\|_2^2 \\ &= 2\sum_{k=1}^{n_p} p_k + \inf_{Q\in\mathbb{R}\mathcal{H}_\infty} \|\mathrm{e}^{s\tau}(1 - Y) + N_0 Q\|_2^2. \end{aligned}$$

Further we have

$$\mathrm{e}^{s\tau}(1 - Y(s)) = -\mathrm{e}^{s\tau}G_1(s) - G_2(s),$$

where

$$\begin{aligned} G_1(s) &:= F(I - \mathrm{e}^{-(sI-A)\tau})(sI - A)^{-1}B, \\ G_2(s) &:= F\mathrm{e}^{A\tau}(sI - A + LC)^{-1}B. \end{aligned}$$

It can be shown that $G_1 \in \mathcal{H}_2^\perp$ and $G_2 \in \mathcal{H}_2$. Furthermore, since $\mathrm{e}^{s\tau}$ is inner and N_0 is minimum phase, then

$$\begin{aligned} \|\mathrm{e}^{s\tau}G_1(s)\|_2^2 &= \|G_1(s)\|_2^2, \\ \inf_{Q\in\mathbb{R}\mathcal{H}_\infty} \|-G_2(s) + N_0 Q\|_2^2 &= 0, \end{aligned}$$

by properly selecting a $Q \in \mathbb{R}\mathcal{H}_\infty$. The proof is completed by fact that

$$\|G_1(s)\|_2^2 = \int_0^\tau F\mathrm{e}^{At}BB^\mathrm{T}\mathrm{e}^{A^\mathrm{T}t}F^\mathrm{T}\,\mathrm{d}t.$$

(See [7] for the details.) □

We provide two direct implication of Theorem 1 regarding the number of unstable poles.

Corollary 1. *If P has only one unstable pole p, then*

$$E^* = 2p\mathrm{e}^{2p\tau}.$$

Proof. For the case of P has a single unstable pole p, we have $A_1 = p$ and $B_1 = 1$. By solving (13) we obtain $F = 2p$. Hence,

$$E^* = 2p + 2p(\mathrm{e}^{2p\tau} - 1) = 2p\mathrm{e}^{2p\tau}.$$

It is proved. □

For a plant with more than one unstable poles it seems hard to obtain a general closed-form expression for E^*. However, we can derive a tight upper bound for it.

Corollary 2. *If P has unstable poles $p_k \in \mathbb{C}_+$ $(k = 1, \ldots, n_p)$, then*

$$\begin{aligned} E^* &= \left(2\sum_{k=1}^{n_p} p_k\right)\mathrm{e}^{2\tau\sum_{k=1}^{n_p} p_k} - \mathcal{O}(\tau^2) \\ &\le \left(2\sum_{k=1}^{n_p} p_k\right)\mathrm{e}^{2\tau\sum_{k=1}^{n_p} p_k}. \end{aligned}$$

Proof. See [7]. □

Example 1. *Suppose that the plant has two real distinct unstable poles p_1 and p_2. Then,*

$$E^* = 2(p_1 + p_2)\mathrm{e}^{2(p_1+p_2)\tau}\gamma,$$

where

$$\gamma = \frac{(p_1\mathrm{e}^{-p_1\tau} - p_2\mathrm{e}^{-p_2\tau})^2 + p_1p_2(\mathrm{e}^{-p_1\tau} - \mathrm{e}^{-p_2\tau})^2}{(p_1 - p_2)^2}.$$

Immediately we have $\lim_{\tau\to 0}\gamma = 1$. Also it is not difficult to verify that whenever $p_1 \to p$ and $p_2 \to p$, i.e., the unstable poles close each other, then

$$\gamma \to \frac{2p^2\tau^2 - 2p\tau + 1}{\mathrm{e}^{2p\tau}}.$$

Fig. 2 depicts the term γ for different values of p_1 and p_2 in the interval $[0,\ 3]$ for $\tau = 1$ and $\tau = 0.3$. While Fig. 3 plots the $\mathcal{O}(\tau^2)$-term with respect to the location of unstable poles. We can see from those two figures that the conservativeness of the bound reduces as $\tau \to 0$.

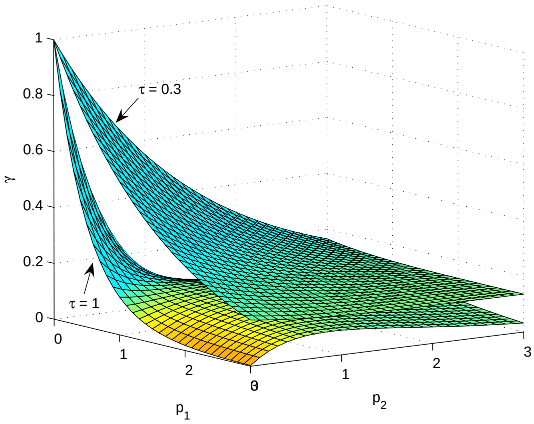

FIGURE 2. Plot of γ versus the location of unstable poles p_1 and p_2 for $\tau = \{1, 0.3\}$.

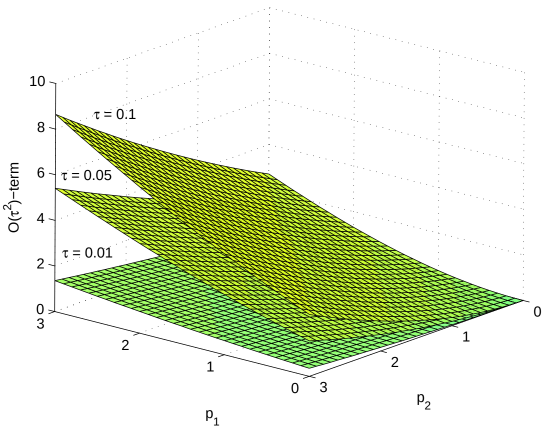

FIGURE 3. Plot of $\mathcal{O}(\tau^2)$-term versus the location of unstable poles p_1 and p_2 for $\tau = \{0.1, 0.05, 0.01\}$.

4. Non-minimum Phase Case

We begin this section by presenting some preliminary results regarding the energy regulation problem of LTI discrete-time and delta domain systems, which are previously studied in [3].

Theorem 2. [3] *Suppose that the discrete-time plant $P_{\mathrm{d}}(z)$ has unstable poles λ_k $(k = 1, \ldots, n_\lambda)$ and non-minimum phase zeros η_i $(i = 1, \ldots, n_\eta)$. Then, the minimal regulation energy E_{d}^* is given by*

$$E_{\mathrm{d}}^* = E_{\mathrm{dm}} + E_{\mathrm{dn}}, \tag{15}$$

where

$$\begin{aligned} E_{\mathrm{dm}} &:= \prod_{k=1}^{n_\lambda} |\lambda_k|^2 - 1, \\ E_{\mathrm{dn}} &:= \sum_{i,j=1}^{n_\eta} \frac{(|\eta_i|^2-1)(|\eta_j|^2-1)}{\bar{b}_i b_j(\bar{\eta}_i\eta_j - 1)}\bar{\beta}_i\beta_j \end{aligned}$$

with

$$\begin{aligned} b_i &:= \begin{cases} 1 & ;\ n_\eta = 1 \\ \displaystyle\prod_{j\neq i} \frac{\eta_i - \eta_j}{\bar{\eta}_j\eta_i - 1} & ;\ n_\eta \geq 2 \end{cases} \\ \beta_i &:= \prod_{k=1}^{n_\lambda} \bar{\lambda}_k - \prod_{k=1}^{n_\lambda} \frac{\bar{\lambda}_k\eta_i - 1}{\eta_i - \lambda_k}. \end{aligned}$$

Corollary 3. *Suppose that the plant $P_{\mathrm{d}}(z)$ has relative degree v, non-minimum phase zeros η_i $(i = 1, \ldots, n_\eta)$ and only one unstable pole λ. Then,*

$$E_{\mathrm{d}}^* = \lambda^{2(v-1)}(\lambda^2 - 1)\left[\prod_{i=1}^{n_\eta} \frac{\lambda\eta_i - 1}{\eta_i - \lambda}\right]^2.$$

Proof. Let the plant $P(z)$ has only one unstable pole λ. In addition, if $P(z)$ has relative degree 1 and one common non-minimum phase zero η, then from the expressions in Theorem 2 we obtain

$$E_{\mathrm{d}}^* = (\lambda^2 - 1)\left(\frac{\lambda\eta - 1}{\eta - \lambda}\right)^2.$$

If $P(z)$ has relative degree 2 and two common non-minimum phase zeros η_1, η_2, then

$$E_{\mathrm{d}}^* = \lambda^2(\lambda^2 - 1)\left(\frac{\lambda\eta_1 - 1}{\eta_1 - \lambda}\frac{\lambda\eta_2 - 1}{\eta_2 - \lambda}\right)^2.$$

Furthermore, if $P(z)$ has relative degree 3 and three common non-minimum phase zeros η_1, η_2, η_3, then

$$E_{\mathrm{d}}^* = \lambda^4(\lambda^2 - 1)\left(\frac{\lambda\eta_1 - 1}{\eta_1 - \lambda}\frac{\lambda\eta_2 - 1}{\eta_2 - \lambda}\frac{\lambda\eta_3 - 1}{\eta_3 - \lambda}\right)^2.$$

In general, if $P(z)$ has relative degree v and common non-minimum phase zeros η_i $(i = 1, \ldots, n_\eta)$, then

$$E_{\mathrm{d}}^* = \lambda^{2(v-1)}(\lambda^2 - 1)\left[\prod_{i=1}^{n_\eta} \frac{\lambda\eta_i - 1}{\eta_i - \lambda}\right]^2.$$

It is proved. □

The delta domain version for the case in Corollary 3 is given as follows.

Corollary 4. *Suppose that the plant $P_T(\delta)$ has relative degree v, non-minimum phase zeros ζ_i $(i = 1, \ldots, n_\zeta)$ and only one unstable pole ρ. Then,*

$$E_T^* = (T\rho + 1)^{2(v-1)}(T\rho^2 + 2\rho)\left[\prod_{i=1}^{n_\zeta} \frac{T\zeta_i\rho + \zeta_i + \rho}{\zeta_i - \rho}\right]^2.$$

Now we consider the delay-time plant (1) where

$$P_0(s) := \frac{P_{\mathrm{n}}(s)}{s - p}$$

with $p \in \mathbb{C}_+$ is the unstable pole of P and P_{n} is rational, stable, and strictly proper transfer function which has non-minimum phase zeros z_i $(i = 1, \ldots, n_z)$. In other words, we consider a non-minimum phase delay-time plant which has only single unstable pole p:

$$P(s) = \frac{P_{\mathrm{n}}(s)}{s - p}\mathrm{e}^{-s\tau}. \tag{16}$$

Under the zero-order hold operations of sampling time T, we then obtain the corresponding delta domain plant of (16) as follows

$$P_T(\delta) = \frac{P_{T\mathrm{n}}(\delta)}{(T\delta + 1)^{\tau/T+1}(\delta - \rho)}, \tag{17}$$

where ρ is the unstable pole of $P_T(\delta)$ and $P_{T\mathrm{n}}$ is stable and has non-minimum phase zeros ζ_i $(i = 1, \ldots, n_\zeta)$. Note that τ/T relative degrees are contributed by the discretization of the delay part $\mathrm{e}^{-s\tau}$, while 1 relative degree is from that of $P_{T\mathrm{n}}(s)$. The optimal performance of (17) is then can be obtained by application of Corollary 4.

Corollary 5. *Suppose the corresponding delta domain of the plant $P_T(\delta)$ is given by (17). Then,*

$$E_T^* = (T\rho + 1)^{2\tau/T}(T\rho^2 + 2\rho)\left[\prod_{i=1}^{n_\zeta} \frac{T\zeta_i\rho + \zeta_i + \rho}{\zeta_i - \rho}\right]^2.$$

By facts that $\rho = (\mathrm{e}^{pT} - 1)/T$ and $\zeta_i = (\mathrm{e}^{z_iT} - 1)/T$, then we immediately have[1]

$$\lim_{T\to 0} E_T^* = 2p\,\mathrm{e}^{2p\tau}\left[\prod_{i=1}^{n_z} \frac{z_i + p}{z_i - p}\right]^2.$$

Hence, by the continuity property we can derive the energy regulation performance for delay-time system (16), as shown in the following result.

[1]It can be shown that the limiting zeros do not give any effect when the sampling time sufficiently small. See [1] for the definition of limiting zeros.

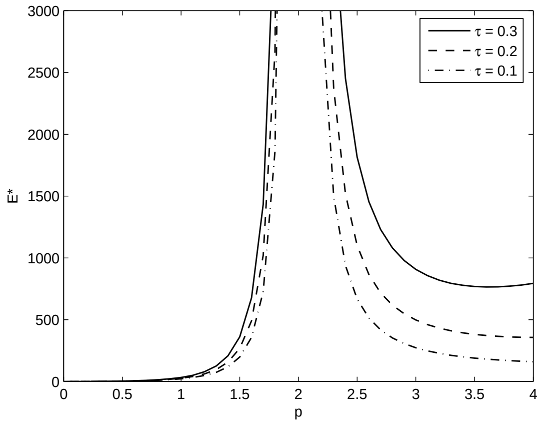

FIGURE 4. Plot of E^* versus the location of unstable pole p for $\tau = \{0.1, 0.2, 0.3\}$.

Proposition 1. *Suppose the plant $P(s)$ is given by (16). Then, the optimal regulation performance is given by*

$$E^* = 2p\,\mathrm{e}^{2p\tau}\left[\prod_{i=1}^{n_z}\frac{z_i+p}{z_i-p}\right]^2. \tag{18}$$

Proposition 1, which extends Corollary 1, admits that unstable pole and non-minimum phase zero which close each other generally worsen the regulation performance. Furthermore, if $z_i = \infty$ then $E^* = 2p\mathrm{e}^{2p\tau}$, which confirms Corollary 1. Additionally if $\tau = 0$ and $n_z = 1$, i.e., we consider an LTI case with single unstable pole p and non-minimum phase zero z, then

$$E^* = 2p\left(\frac{z+p}{z-p}\right)^2,$$

which can be confirmed either by [10, Theorem 2] or [12, Proposition 3.1].

Example 2. *We illustrate the result by picking one simple example, where we consider the following delay-time plant:*

$$P(s) = \frac{s-2}{s-p}\mathrm{e}^{-s\tau}, \quad p > 0.$$

Fig. 4 plots the optimal regulation performance E^*, which is computed by using (18), for different value of p in the interval $(0, 4]$ *and different value of τ.*

5. CONCLUSION

In the present paper we have studied the energy regulation problem of delay-time systems. We have provided the analytical closed-form expression of the optimal performance in terms of the plant properties.

In general, time-delay gives its effects in exponential way as well as the unstable pole of the plant. Whenever the plant is non-minimum phase, the unstable pole and non-minimum phase zero which close each other will deteriorate the regulation performance.

References

[1] K.J. Åström, P. Hagander, and J. Sternby, "Zeros of sampled systems," *Automatica*, vol. 20, no. 1, pp. 31–38, Jan. 1984.

[2] T. Bakhtiar and S. Hara, "$\mathcal{H}_2$ regulation performance limitations for unstable/non-minimum phase SIMO systems." In *Proc. of the 34rd SICE Symposium on Control Theory*, Osaka, Japan, Oct.–Nov. 2005, pp. 489–492.

[3] T. Bakhtiar and S. Hara, "$\mathcal{H}_2$ control performance limitations for SIMO systems: a unified approach," in *Proc. 6th Asian Control Conference (ASCC2006)*, Bali, Indonesia, July 2006, pp. 555–563.

[4] T. Bakhtiar and S. Hara, "$\mathcal{H}_2$ regulation performance limits for SIMO feedback control systems." In *Proc. 7th International Symposium on Mathematical Theory of Networks and Systems (MTNS2006)*, Kyoto, Japan, July 2006, pp. 1966–1973.

[5] H.W. Bode, *Network Analysis and Feddback Amplifier Design*, Princeton, NJ: Van Nostrand, 1945.

[6] J.H. Braslavsky, R.H. Middleton, and J. Freudenberg, "Feedback stabilization over signal-to-noise ratio constrained channels," in *Proc. 2004 American Control Conference*, Boston, USA, June–July 2004, pp. 4903–4909.

[7] J.H. Braslavsky, R.H. Middleton, and J.S. Freudenberg, "Effects of time delay on feedback stabilization over signal-to-noise ratio constrained channels," in *Proc. 16th IFAC World Congress*, Prague, July 2005.

[8] J. Chen, S. Hara, and G. Chen, "Best tracking and regulation performance under control energy constraint," *IEEE Trans. Automat. Contr.*, vol. 48, no. 8, pp. 1320–1336, Aug. 2003.

[9] G.H. Golub and C.F. Van Loan (1996). *Matrix Computations*, 3rd Edition. The John Hopkins University Press.

[10] S. Hara and C. Kogure, "Relationship between H_2 control performance limits and RHP pole/zero locations," in *Proc. SICE Annual Conference*, Fukui, Japan, Aug. 2003, pp. 1242–1246.

[11] L.W. Johnson and R.D. Riess (1982). *Numerical Analysis*, 2nd Edition. Reading, Mass.: Addison-Wesley.

[12] R. Middleton, J.H. Braslavsky, and J. Freudenberg, "Stabilization of non-minimum phase plants over signal-to-noise ratio constrained channel," in *Proc. 5th Asian Control Conference*, Melbourne, Australia, July 2004.

[13] R.H. Middleton and G.C. Goodwin (1990). *Digital control and estimation: a unified approach*, Englewood Cliffs, N.J.: Prentice-Hall.