ASYMPTOTIC NORMALITY OF A KERNEL-TYPE ESTIMATOR FOR THE INTENSITY OF A PERIODIC POISSON PROCESS

I WAYAN MANGKU

Department of Mathematics,
Faculty of Mathematics and Natural Sciences,
Bogor Agricultural University
Jl. Meranti, Kampus IPB Darmaga, Bogor, 16680 Indonesia

Abstract. In this paper we prove asymptotic normality of a kernel type estimator for the intensity of a periodic Poisson process. This paper is a continuation of [10]. As in [10], we consider the situation when the period is known in order to be able to present simple proofs of the results.

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1. Introduction

This paper is a continuation of [10]. We consider kernel type estimation of the intensity function λ at a given point $s \in [0, \infty]$, using only a single realization $N(\omega)$ of the periodic Poisson process N observed in $[0, n]$. This problem arises frequently in many diverse areas including communications, hydrology, meteorology, insurance, reliability, medical sciences, seismology, and some others (cf. also [1], [2], [3], [4], [7], [8], [9], [11], [12])

Let N be a Poisson process on $[0, \infty)$ with (unknown) locally integrable intensity function λ. We assume that λ is a periodic function with (known) period τ. We do not assume any parametric form of λ, except that it is periodic. That is, for each point $s \in [0, \infty)$ and all $k \in \mathbf{Z}$, with $\mathbf{Z}$ denotes the set of integers, we have

$$\lambda(s + k\tau) = \lambda(s). \tag{1.1}$$

Suppose that, for some $\omega \in \Omega$, a single realization $N(\omega)$ of the Poisson process N defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with intensity function λ is observed, though only within a bounded interval $[0, n]$. Our goal in this paper is to study asymtotic properties and to prove asymptotic normality of a kernel-type estimator for λ at a given point $s \in [0, \infty)$ using only a single realization $N(\omega)$ of the Poisson process N observed in interval $[0, n]$.

Throughout this paper, we will assume that s is a Lebesgue point of λ, that is we have

$$\lim_{h \downarrow 0} \frac{1}{2h} \int_{-h}^{h} |\lambda(s+x) - \lambda(s)| dx = 0 \tag{1.2}$$

(eg. see [14], p.107-108).

Since λ is a periodic function with period τ, the problem of estimating λ at a given point $s \in [0, \infty)$ can be reduced into a problem of estimating λ at a given point $s \in [0, \tau)$. Hence, for the rest of this paper, we will assume that $s \in [0, \tau)$.

2. The estimator and some results

Let $K : \mathbf{R} \to \mathbf{R}$ be a real valued function, called *kernel*, which satisfies the following conditions: (K1) K is a probability density function, (K2) K is bounded, and (K3) K has (closed) support $[-1, 1]$. Let also h_n be a sequence of positive real numbers converging to 0, that is,

$$h_n \downarrow 0, \tag{2.1}$$

as $n \to \infty$. Now, we may define the estimator of λ at a given point $s \in [0, \tau)$ as follows

$$\hat{\lambda}_{n,K}(s) := \frac{\tau}{n} \sum_{k=0}^{\infty} \frac{1}{h_n} \int_0^n K\left(\frac{x - (s + k\tau)}{h_n}\right) N(dx). \tag{2.2}$$

This estimator is a special case of a more general kernel-type estimator of the intensity of a periodic Poisson process, which includes the case when the period τ has to be estimated (see Helmers, Mangku and Zitikis ([5], [6]).

First we survey some results on statistical properties of kernel-type estimator for the intensity of a periodic Poisson process for the case when the period is known, which are given in the following theorems. We refer to [6] for the case when the period is unknown.

Theorem 2.1. (Asymptotic approximation to the bias)

Suppose that the intensity function λ is periodic and locally integrable, and has finite second derivative λ'' at s. If the kernel K is symmetric and satisfies conditions $(K1), (K2), (K3)$, and h_n satisfies assumptions (2.1) and $nh_n^2 \to \infty$, then

$$\mathbf{E}\hat{\lambda}_{n,K}(s) = \lambda(s) + \frac{1}{2}\lambda''(s) h_n^2 \int_{-1}^{1} x^2 K(x) dx + o(h_n^2), \tag{2.3}$$

as $n \to \infty$.

Theorem 2.2. (Asymptotic approximation to the variance)

Suppose that the intensity function λ is periodic and locally integrable. If the kernel K satisfies conditions $(K1),(K2),(K3)$, and h_n satisfies assumptions (2.1), then

$$Var\left(\hat{\lambda}_{n,K}(s)\right) = \frac{\tau\lambda(s)}{nh_n}\int_{-1}^{1} K^2(x)dx + o\left(\frac{1}{nh_n}\right) \tag{2.4}$$

as $n \to \infty$, provided s is a Lebesgue point of λ.

From Theorems 2.1 and 2.2 we obtain an asymptotic approximation to the *Mean-Squared-Error* (MSE) of $\hat{\lambda}_{n,K}(s)$, which is given in the following corollary.

Corollary 2.3. (Asymptotic approximation to MSE)

Suppose that the intensity function λ is periodic and locally integrable, and has finite second derivative λ'' at s. If the kernel K is symmetric and satisfies conditions $(K1),(K2),(K3)$, and h_n satisfies assumptions (2.1) and $nh_n^2 \to \infty$, then

$$\begin{aligned} MSE(\hat{\lambda}_{n,K}(s)) &= \frac{\tau\lambda(s)}{nh_n}\int_{-1}^{1} K^2(x)dx \\ &+ \frac{1}{4}\left(\lambda''(s)\int_{-1}^{1} x^2K(x)dx\right)^2 h_n^4 + o\left(\frac{1}{nh_n}\right) + o(h_n^4), \end{aligned} \tag{2.5}$$

as $n \to \infty$.

Now, we consider the r.h.s. of (2.5). By minimizing the sum of its first and second terms (the main terms for the variance and the squared bias), we then get the optimal choice of h_n, which is given by

$$h_n = \left[\frac{\tau\lambda(s)\int_{-1}^{1} K^2(x)dx}{\left(\lambda''(s)\int_{-1}^{1} x^2K(x)dx\right)^2}\right]^{\frac{1}{5}} n^{-\frac{1}{5}}. \tag{2.6}$$

With this choice of h_n, the optimal rate of decrease of $MSE(\hat{\lambda}_{n,K}(s))$ is of order $\mathcal{O}(n^{-4/5})$ as $n \to \infty$.

The main result of this paper is the following theorem.

Theorem 2.4. (Asymptotic normality)

Suppose that the intensity function λ is periodic and locally integrable, and has finite second derivative λ'' at s. Let the kernel K is symmetric and satisfies conditions $(K1),(K2),(K3)$. Let also the bandwidth h_n satisfies assumptions (2.1) and $nh_n^2 \to \infty$, as $n \to \infty$.
(i) If $\sqrt{nh_n^5} \to 1$, as $n \to \infty$, then

$$\sqrt{nh_n}\left(\hat{\lambda}_{n,K}(s) - \lambda(s)\right) \xrightarrow{d} Normal\left(\mu, \sigma^2\right), \tag{2.7}$$

as $n \to \infty$*, with* $\mu = \frac{1}{2}\lambda''(s)\int_{-1}^{1} x^2 K(x)dx$ *and* $\sigma^2 = \tau\lambda(s)\int_{-1}^{1} K^2(x)dx$.
(ii) If $\sqrt{nh_n^5} \to 0$*, as* $n \to \infty$*, then*

$$\sqrt{nh_n}\left(\hat{\lambda}_{n,K}(s) - \lambda(s)\right) \xrightarrow{d} Normal\left(0, \sigma^2\right), \tag{2.8}$$

as $n \to \infty$.

3. Simulations

For the simulations, we consider the intensity function

$$\lambda(s) = A \exp\left\{\rho \cos\left(\frac{2\pi s}{\tau} + \phi\right)\right\},$$

that is the intensity function discussed in Vere-Jones [13]. We chose $A = 2$, $\rho = 1$, $\tau = 5$ and $\phi = 0$. With this choice of the parameters, we have

$$\lambda(s) = 2\exp\left\{\cos\left(\frac{2\pi s}{\tau}\right)\right\}. \tag{3.1}$$

This function achieves its maximum $2e = 5.4366$ at $s = 5k$ and its minimum $2e^{-1} = 0.7358$ at $s = 2.5 + 5k$, for any integer k. Since λ is periodic with period τ, the problem of estimating λ at a given $s \in [0, \infty)$ can be reduced to the problem of estimating λ at a given $s \in [0, \tau)$. In our simulations we consider three values of s, namely $s = 2.6$ (a small value of $\lambda(s)$), $s = 4.0$ (a moderate value of $\lambda(s)$) and $s = 4.9$ (a large value of $\lambda(s)$). We use $[0, n] = [0, 1000]$ and take the 'optimal' choice for the bandwidth h_n.

In this example, we use the estimator given by (2.2) with kernel $K = \frac{1}{2}\mathbf{I}([-1, 1])$, that is

$$\hat{\lambda}_n(s) := \frac{\tau}{n}\sum_{k=0}^{\infty} \frac{N([s + k\tau - h_n, s + k\tau + h_n] \cap [0, n])}{2h_n}.$$

Asymptotic approximations to the variance and bias of this estimator are given respectively by (cf. Theorems 2.1 and 2.2)

$$Bias\left(\hat{\lambda}_n(s)\right) = \frac{\lambda''(s)}{6}h_n^2 + o(h_n^2), \tag{3.2}$$

and

$$Var\left(\hat{\lambda}_n(s)\right) = \frac{\lambda(s)\tau}{2nh_n} + o\left(\frac{1}{nh_n}\right), \tag{3.3}$$

as $n \to \infty$. The optimal choice for the bandwidth h_n is given by (cf. (2.6))

$$h_n = \left[\frac{9\lambda(s)\tau}{(2(\lambda''(s))^2)}\right]^{1/5} (n)^{-1/5}. \tag{3.4}$$

Result of the simulation is as follows.

(i) For $s = 2.6$, we have $\lambda(s) = 0.7416$ and $\lambda''(s) = 1.1802$. By (3.4), with $n = 1000$ and $\tau = 5$, we obtain the (optimal) choice of bandwidth $h_n = 0.4128$. By (3.3) and (3.2), we obtain the numerical values of the asymptotic approximations to respectively the variance and the bias of $\hat{\lambda}_n(s)$: $Var(\hat{\lambda}_n(s)) = 0.0045$ and $Bias(\hat{\lambda}_n(s)) = 0.0335$. From the simulation, using $M = 10^4$ independent realizations of the process N observed in the $[0,n] = [0,1000]$, we obtain respectively $\hat{V}ar(\hat{\lambda}_n(s)) = 0.0047$ and $\hat{B}ias(\hat{\lambda}_n(s)) = 0.0303$, where $\hat{V}ar(\hat{\lambda}_n(s))$ is the sample variance $\frac{1}{M-1}\sum_{i=1}^{M}(\hat{\lambda}_{n,i}(s) - \frac{1}{M}\sum_{j=1}^{M}\hat{\lambda}_{n,j}(s))^2$ and $\hat{B}ias(\hat{\lambda}_n(s))$ is the sample mean $M^{-1}\sum_{j=1}^{M}\hat{\lambda}_{n,j}(s)$ minus $\lambda(s)$. Summarizing, we have
$Var(\hat{\lambda}_n(s)) - \hat{V}ar(\hat{\lambda}_n(s)) = 0.0045 - 0.0047 = -0.0002$ and
$Bias(\hat{\lambda}_n(s)) - \hat{B}ias(\hat{\lambda}_n(s)) = 0.0335 - 0.0303 = 0.0032$.

(ii) For $s = 4.0$, we have $\lambda(s) = 2.7242$ and $\lambda''(s) = 2.5617$.
By (3.4), we obtain $h_n = 0.3927$. By (3.3) and (3.2) and from the simulation ($M = 10^4$) we obtain
$Var(\tilde{\lambda}_n(s)) - \hat{V}ar(\hat{\lambda}_n(s)) = 0.0173 - 0.0178 = -0.0005$ and
$Bias(\hat{\lambda}_n(s)) - \hat{B}ias(\hat{\lambda}_n(s)) = 0.0658 - 0.0472 = 0.0186$.

(iii) For $s = 4.9$, we have $\lambda(s) = 5.3939$ and $\lambda''(s) = -8.3167$.
By (3.4), we obtain $h_n = 0.2811$. By (3.3) and (3.2) and from the simulation ($M = 10^4$) we obtain
$Var(\hat{\lambda}_n(s)) - \hat{V}ar(\hat{\lambda}_n(s)) = 0.0480 - 0.0462 = 0.0018$ and
$Bias(\hat{\lambda}_n(s)) - \hat{B}ias(\hat{\lambda}_n(s)) = -0.1095 - (-0.1404) = 0.0309$.

In this example we find that, the asymptotic approximations to respectively the variance and bias of the estimator given in (2.2) for the uniform kernel are relatively close to the numerical values obtained in the simulation. So we can conclude that the first order asymptotics derived in (2.3) and (2.4) work well in approximating the bias and the variance in finite samples.

4. Proof of Theorem 2.1

Note that

$$\begin{aligned}
\mathbf{E}\hat{\lambda}_{n,K}(s) &= \frac{\tau}{n}\sum_{k=0}^{\infty}\frac{1}{h_n}\int_0^n K\left(\frac{x-(s+k\tau)}{h_n}\right)\mathbf{E}N(dx) \\
&= \frac{\tau}{n}\sum_{k=0}^{\infty}\frac{1}{h_n}\int_0^n K\left(\frac{x-(s+k\tau)}{h_n}\right)\lambda(x)dx \\
&= \frac{\tau}{n}\sum_{k=0}^{\infty}\frac{1}{h_n}\int_{\mathbf{R}} K\left(\frac{x-(s+k\tau)}{h_n}\right)\lambda(x)\mathbf{I}(x\in[0,n])dx.
\end{aligned} \tag{4.1}$$

By a change of variable and using (1.1), we can write the r.h.s. of (4.1) as

$$\begin{aligned}
& \frac{\tau}{n}\sum_{k=0}^{\infty}\frac{1}{h_n}\int_{\mathbf{R}} K\left(\frac{x}{h_n}\right)\lambda(x+s+k\tau)\mathbf{I}(x+s+k\tau\in[0,n])dx \\
= & \frac{\tau}{n}\sum_{k=0}^{\infty}\frac{1}{h_n}\int_{\mathbf{R}} K\left(\frac{x}{h_n}\right)\lambda(x+s)\mathbf{I}(x+s+k\tau\in[0,n])dx \\
= & \frac{\tau}{nh_n}\int_{\mathbf{R}} K\left(\frac{x}{h_n}\right)\lambda(x+s)\sum_{k=0}^{\infty}\mathbf{I}(x+s+k\tau\in[0,n])dx. \qquad (4.2)
\end{aligned}$$

Now note that

$$\sum_{k=0}^{\infty}\mathbf{I}(x+s+k\tau\in[0,n])\in\left[\frac{n}{\tau}-1,\frac{n}{\tau}-1\right]. \qquad (4.3)$$

Then, the r.h.s. of (4.2) can be written as

$$\begin{aligned}
& \frac{\tau}{n\ h_n}\int_{\mathbf{R}} K\left(\frac{x}{h_n}\right)\lambda(x+s)\left(\frac{n}{\tau}+\mathcal{O}(1)\right)dx \\
= & \frac{1}{h_n}\int_{\mathbf{R}} K\left(\frac{x}{h_n}\right)\lambda(x+s)dx+\mathcal{O}\left(\frac{1}{n}\right) \\
= & \int_{-1}^{1} K(x)\lambda(s+xh_n)dx+\mathcal{O}\left(\frac{1}{n}\right), \qquad (4.4)
\end{aligned}$$

as $n\to\infty$. By the Young's form of Taylor's theorem, we have

$$\lambda(s+xh_n)=\lambda(s)+\frac{\lambda'(s)}{1!}xh_n+\frac{\lambda''(s)}{2!}x^2h_n^2+o(h_n^2), \qquad (4.5)$$

jika $n\to\infty$. Substituting (4.5) into the r.h.s. of (4.4), we obtain

$$\begin{aligned}
\mathbf{E}\hat{\lambda}_{n,K}(s) &= \int_{-1}^{1} K(x)\left(\lambda(s)+\frac{\lambda'(s)}{1!}xh_n+\frac{\lambda''(s)}{2!}x^2h_n^2\right)dx \\
&\quad +o(h_n^2)+\mathcal{O}\left(\frac{1}{n}\right) \\
&= \lambda(s)\int_{-1}^{1} K(x)dx+\lambda'(s)h_n\int_{-1}^{1} xK(x)dx \\
&\quad +\frac{\lambda''(s)h_n^2}{2}\int_{-1}^{1} x^2K(x)dx+o(h_n^2)+\mathcal{O}\left(\frac{1}{n}\right), \qquad (4.6)
\end{aligned}$$

as $n\to\infty$. By assumption (K1) and (K3) we have $\int_{-1}^{1}K(x)dx=1$. Since the kernel K is symmetric, an easy calculation shows that the second term on the r.h.s. of (4.6) is equal to zero. By the assumption $nh_n^2\to\infty$, we have the last term on the r.h.s. of (4.6) is of order $o(h_n^2)$, as $n\to\infty$. Hence we obtain (2.3). This completes the proof of Theorem 2.1.

5. Proof of Theorem 2.2

The variance of $\hat{\lambda}_{n,K}(s)$ can be computed as follows

$$Var\left(\hat{\lambda}_{n,K}(s)\right) = \frac{\tau^2}{n^2} Var\left(\sum_{k=0}^{\infty} \frac{1}{h_n} \int_0^n K\left(\frac{x-(s+k\tau)}{h_n}\right) N(dx)\right). \tag{5.1}$$

By (2.1), for sufficiently large n, we have that the intervals $[s + k\tau - h_n, s + k\tau + h_n]$ and $[s + j\tau - h_n, s + j\tau + h_n]$ are not overlap for all $k \neq j$. This implies, for all $k \neq j$,

$$K\left(\frac{x-(s+k\tau)}{h_n}\right) N(dx) \text{ and } K\left(\frac{x-(s+j\tau)}{h_n}\right) N(dx)$$

are independent. Hence, the r.h.s. of (5.1) can be computed as follows

$$\begin{aligned}
&\frac{\tau^2}{n^2h_n^2} \sum_{k=0}^{\infty} \int_0^n K^2\left(\frac{x-(s+k\tau)}{h_n}\right) Var(N(dx)) \\
= &\frac{\tau^2}{n^2h_n^2} \sum_{k=0}^{\infty} \int_0^n K^2\left(\frac{x-(s+k\tau)}{h_n}\right) \mathbf{E}N(dx) \\
= &\frac{\tau^2}{n^2h_n^2} \sum_{k=0}^{\infty} \int_0^n K^2\left(\frac{x-(s+k\tau)}{h_n}\right) \lambda(x)dx. \qquad (5.2)
\end{aligned}$$

By a change of variable and using (1.1), the r.h.s. of (5.2) can be written as

$$\begin{aligned}
&\frac{\tau^2}{n^2h_n^2} \sum_{k=0}^{\infty} \int_{\mathbf{R}} K^2\left(\frac{x}{h_n}\right) \lambda(x+s+k\tau)\mathbf{I}(x+s+k\tau \in [0,n])dx \\
= &\frac{\tau^2}{n^2h_n^2} \sum_{k=0}^{\infty} \int_{\mathbf{R}} K^2\left(\frac{x}{h_n}\right) \lambda(x+s)\mathbf{I}(x+s+k\tau \in [0,n])dx. \qquad (5.3)
\end{aligned}$$

The r.h.s. of (5.3) is equal to

$$\begin{aligned}
&\frac{\tau^2}{n^2h_n^2} \int_{\mathbf{R}} K^2\left(\frac{x}{h_n}\right) (\lambda(x+s) - \lambda(s)) \sum_{k=0}^{\infty} \mathbf{I}(x+s+k\tau \in [0,n])dx \\
&+\frac{\lambda(s)\tau^2}{n^2h_n^2} \int_{\mathbf{R}} K^2\left(\frac{x}{h_n}\right) \sum_{k=0}^{\infty} \mathbf{I}(x+s+k\tau \in [0,n])dx. \qquad (5.4)
\end{aligned}$$

By (4.3), the quantity in (5.4) can be written as

$$\begin{aligned}
&\frac{\tau^2}{n^2h_n^2} \int_{\mathbf{R}} K^2\left(\frac{x}{h_n}\right) (\lambda(x+s) - \lambda(s)) \left(\frac{n}{\tau} + \mathcal{O}(1)\right) dx \\
&+\frac{\lambda(s)\tau^2}{n^2h_n^2} \int_{\mathbf{R}} K^2\left(\frac{x}{h_n}\right) \left(\frac{n}{\tau} + \mathcal{O}(1)\right) dx. \qquad (5.5)
\end{aligned}$$

Since the kernel K is bounded and has support in $[-1, 1]$, by (1.2), we see that the first term on the r.h.s. of (5.5) is of order $o(n^{-1}(h_n)^{-1}))$, as $n \to \infty$. By a change of variable, the second term on the r.h.s. of (5.5) can be written as

$$\begin{aligned} & \frac{\lambda(s)\tau^2}{n^2 h_n} \int_{-1}^{1} K^2(x) \left(\frac{n}{\tau} + \mathcal{O}(1) \right) dx \\ = & \frac{\lambda(s)\tau}{n h_n} \int_{-1}^{1} K^2(x) dx + \mathcal{O}\left(\frac{1}{n^2 h_n} \right), \end{aligned} \tag{5.6}$$

as $n \to \infty$. Since the second term on the r.h.s. of (5.6) is of order $o(n^{-1}(h_n)^{-1}))$, as $n \to \infty$, we obtain (2.4). This completes the proof of Theorem 2.2.

6. Proof of Theorem 2.4

First we write the l.h.s. of (2.7) and (2.8) as

$$\sqrt{nh_n}\left(\hat{\lambda}_{n,K}(s) - \mathbf{E}\hat{\lambda}_{n,K}(s)\right) + \sqrt{nh_n}\left(\mathbf{E}\hat{\lambda}_{n,K}(s) - \lambda(s)\right). \tag{6.1}$$

Hence, to prove this theorem, it suffices to show

$$\sqrt{nh_n}\left(\hat{\lambda}_{n,K}(s) - \mathbf{E}\hat{\lambda}_{n,K}(s)\right) \xrightarrow{d} Normal\left(0, \sigma^2\right), \tag{6.2}$$

as $n \to \infty$; if $\sqrt{nh_n^5} \to 1$ as $n \to \infty$, then

$$\sqrt{nh_n}\left(\mathbf{E}\hat{\lambda}_{n,K}(s) - \lambda(s)\right) \to \frac{1}{2}\lambda''(s) \int_{-1}^{1} x^2 K(x) dx, \tag{6.3}$$

as $n \to \infty$; and if $\sqrt{nh_n^5} \to 0$ as $n \to \infty$, then

$$\sqrt{nh_n}\left(\mathbf{E}\hat{\lambda}_{n,K}(s) - \lambda(s)\right) \to 0, \tag{6.4}$$

as $n \to \infty$.

First we consider (6.2). The l.h.s. of (6.2) can be written as

$$\sqrt{nh_n}\sqrt{Var(\hat{\lambda}_{n,K}(s))}\left(\frac{\hat{\lambda}_{n,K}(s) - \mathbf{E}\hat{\lambda}_{n,K}(s)}{\sqrt{Var(\hat{\lambda}_{n,K}(s))}}\right). \tag{6.5}$$

Then, to prove (6.2), it suffices to check

$$\left(\frac{\hat{\lambda}_{n,K}(s) - \mathbf{E}\hat{\lambda}_{n,K}(s)}{\sqrt{Var(\hat{\lambda}_{n,K}(s))}}\right) \xrightarrow{d} Normal\left(0, 1\right), \tag{6.6}$$

and

$$\sqrt{nh_n}\sqrt{Var(\hat{\lambda}_{n,K}(s))} \to \sqrt{\tau\lambda(s)\int_{-1}^{1} K^2(x) dx}, \tag{6.7}$$

as $n \to \infty$. To prove (6.6) we argue as follows. Let, for each $k = 0, 1, 2, \ldots$

$$X_k = \int_0^n K\left(\frac{x - (s + k\tau)}{h_n}\right) N(dx).$$

Since $h_n \downarrow 0$, as $n \to \infty$, then for sufficiently large n, the interval $[s + k\tau - h_n, s + k\tau + h_n]$ and $[s + j\tau - h_n, s + j\tau + h_n]$ for all $k \neq j$ are disjoint. This implies, for all $k \neq j$, the random variables X_k and X_j are independent. Furthermore we have $\{X_k\}$, $k = 0, 1, 2, \ldots$ is a sequence of independent and identically distributed (i.i.d) random variables, having expected value

$$\mathbf{E}X_k = \int_0^n K\left(\frac{x - (s + k\tau)}{h_n}\right)\lambda(x)dx$$

and variance

$$Var(X_k) = \int_0^n K^2\left(\frac{x - (s + k\tau)}{h_n}\right)\lambda(x)dx$$

which are finite, due to the support of the kernel K is the interval $[-1, 1]$. Hence, we can write the estimator $\hat{\lambda}_{n,K}(s)$ as

$$\hat{\lambda}_{n,K}(s) := \frac{\tau}{nh_n}\sum_{k=0}^{\infty} X_k,$$

which is sum of i.i.d. random variables multiplied by a constant. Then, by the Central Limit Theorem, we obtain (6.6). To prove (6.7), we note that the l.h.s. of (6.7) can be written as

$$\sqrt{nh_n Var(\hat{\lambda}_{n,K}(s))}$$

which, by Theorem 2.2 (cf. (2.4)), is equal to

$$\sqrt{\tau\lambda(s)\int_{-1}^{1} K^2(x)dx + o(1)} = \sqrt{\tau\lambda(s)\int_{-1}^{1} K^2(x)dx} + o(1),$$

as $n \to \infty$. Hence we have (6.7).

Next we prove (6.3) and (6.4). By Theorem 2.1 (cf. (2.3)), we have

$$\begin{aligned}\sqrt{nh_n}\left(\mathbf{E}\hat{\lambda}_{n,K}(s) - \lambda(s)\right) &= \frac{1}{2}\lambda''(s)\int_{-1}^{1} x^2K(x)dx\sqrt{nh_n^5} \\ &\quad + o\left(\sqrt{nh_n^5}\right), \qquad (6.8)\end{aligned}$$

as $n \to \infty$. By the assumption $\sqrt{nh_n^5} \to 1$, as $n \to \infty$, we obtain (6.3), while by the assumption $\sqrt{nh_n^5} \to 0$, as $n \to \infty$, we obtain (6.4). This completes the proof of Theorem 2.4.

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