PENDUGAAN PARAMETER DERET WAKTU *HIDDEN* MARKOV HAMILTON *

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ABSTRAK. Pendugaan parameter untuk model deret waktu *Hidden* Markov Hamilton (1994) dilakukan mengunakan Metode *Maximum Likelihood* dan pendugaan ulang menggunakan metode *Expectation Maximization*. Dari kajian ini diperoleh algoritma untuk menduga parameter model.

Kata kunci: Rantai Markov, *Hidden* Markov, Deret waktu *Hidden* Markov, Metode *Expectation Maximization*.

1. PENDAHULUAN

Tulisan ini merupakan kajian tentang pendugaan parameter untuk deret waktu *Hidden* Markov, khususnya model Hamilton (1994). Pendugaan parameter menggunakan metode *Maximum Likelihood* dan pendugaan ulangnya menggunakan metode *Expectatition Maximization* (Metode EM). Dari kedua metode tersebut kemudian diturunkan suatu algoritma yang dapat dipakai secara umum untuk menduga parameter model deret waktu *Hidden* Markov Hamilton (1994).

Tulisan ini dimulai dengan definisi model deret waktu *Hidden* Markov beserta sifat-sifatnya. Pada bagian 3 dibahas Pendugaan Parameter model dan terakhir pada bagian 4 dibahas pendugaan ulang parameter dan algoritmanya.

*Tulisan ini merupakan bagian dari hasil penelitian yang didanai oleh Hibah Penelitian PHK A2 Departemen Matematika IPB tahun 2005 dan 2006

2. MODEL DERET WAKTU *HIDDEN* MARKOV

Pasangan proses stokastik $\{(X_k, Y_k) : k \in \aleph\}$ yang terdefinisi pada ruang peluang (Ω, F, P) dan mempunyai nilai pada $S \times \mathrm{Y}$ disebut model *hidden* Markov apabila $\{X_k\}$ adalah rantai Markov dengan *state* berhingga dan diasumsikan bahwa rantai Markov $\{X_k\}$ tidak diamati. Sehingga $\{X_k\}$ tersembunyi (*hidden*) di balik proses observasi $\{Y_k\}$. Banyaknya elemen dari *S* disebut ukuran (orde) dari model *hidden* Markov.

Pada bagian ini dibahas model *hidden* Markov Hamilton (1994) yang merupakan deret waktu yang memenuhi persamaan

$$Y_t = c(X_t) + \phi(X_t)Y_{t-1} + \varepsilon_t \tag{2.1}$$

di mana:

- $\{X_t\}$ adalah rantai Markov dengan ruang *state* $S = \{1, 2\}$ dan $\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$ merupakan matriks peluang transisinya, di mana $p_{ij} = P\{X_t = j \mid X_{t-1} = i\}$.
- $\{Y_t\}$ adalah proses yang diamati dan bernilai skalar.
- $\{\varepsilon_t\}$ adalah barisan peubah acak yang saling bebas dan berdistribusi normal $N(0, \sigma^2)$.
- $c = (c_1, c_2)$ dan $\phi = (\phi_1, \phi_2) \in \Re^2$, dengan c_1, c_2, ϕ_1, dan ϕ_2 adalah konstanta real.
- $c(X_t) = c_{X_t}$ dan $\phi(X_t) = \phi_{X_t}$.

Jadi model dicirikan oleh parameter $\theta = \left(c_1, c_2, \phi_1, \phi_2, \sigma^2, \mathbf{P}\right)$. Dengan menggunakan metode EM akan diduga parameter $\theta = \left(c_1, c_2, \phi_1, \phi_2, \sigma^2, \mathbf{P}\right)$ dari data *Y*.

Karena $\varepsilon_t \sim N\left(0, \sigma^2\right)$ bebas stokastik identik maka dapat diperoleh fungsi sebaran bagi ε_t sebagai berikut.

$$\begin{aligned} F_{\varepsilon_t}(y_t) = P(\varepsilon_t \le y_t) &= \int_0^{y_t} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{\frac{-(\varepsilon_t - 0)^2}{2\sigma^2}\right\} d\varepsilon_t \\ &= \int_0^{y_t} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{\frac{-(\varepsilon_t)^2}{2\sigma^2}\right\} d\varepsilon_t \,. \end{aligned} \tag{2.2}$$

Berdasarkan persamaan (2.2) diturunkan fungsi sebaran bagi Y_t dengan cara sebagai berikut.

$$
\begin{aligned}
F_{Y_t}(y_t) &= P(Y_t \le y_t) = P\left(c_{X_t} + \phi_{X_t} y_{t-1} + \varepsilon_t \le y_t\right) \\
&= P\left(\varepsilon_t \le y_t - c_{X_t} - \phi_{X_t} y_{t-1}\right) \\
&= \int_0^{y_t - c_{X_t} - \phi_{X_t} y_{t-1}} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{\frac{-(\varepsilon_t)^2}{2\sigma^2}\right\} d\varepsilon_t .
\end{aligned}
$$

Misalkan $v = y_t - c_{X_t} - \phi_{X_t} y_{t-1}$, maka

$$
F_{Y_t}(y_t) = \int_0^{v} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{\frac{-(\varepsilon_t)^2}{2\sigma^2}\right\} d\varepsilon_t
$$

dan

$$
\begin{aligned}
f_{Y_t}(y_t) &= \frac{\partial}{\partial y_t} F_{Y_t}(y_t) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{\frac{-(v)^2}{2\sigma^2}\right\} \cdot \frac{\partial v}{\partial y_t} \\
&= \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{\frac{-\left(y_t - c_{X_t} - \phi_{X_t} y_{t-1}\right)^2}{2\sigma^2}\right\} \cdot 1 \\
&= \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{\frac{-\left(y_t - c_{X_t} - \phi_{X_t} y_{t-1}\right)^2}{2\sigma^2}\right\} .
\end{aligned}
\tag{2.3}
$$

Misalkan Y_t adalah medan-σ yang lengkap dan dibangun oleh $Y_1, Y_2, Y_3 \cdots, Y_t$. Karena X_t merupakan rantai Markov 2 *state* maka terdapat 2 fungsi kerapatan peluang bagi Y_t. Kumpulan fungsi kerapatan tersebut dalam vektor (2×1) dilambangkan dengan η_t. Sehingga diperoleh:

$$
\begin{aligned}
\eta_t &= \begin{bmatrix} f\left(y_t \middle| X_t = 1, \mathrm{Y}_{t-1}; \theta\right) \\ f\left(y_t \middle| X_t = 2, \mathrm{Y}_{t-1}; \theta\right) \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{1}{\sqrt{2\pi}\sigma} \exp\left\{\dfrac{-\left(y_t - c_1 - \phi_1 y_{t-1}\right)^2}{2\sigma^2}\right\} \\ \dfrac{1}{\sqrt{2\pi}\sigma} \exp\left\{\dfrac{-\left(y_t - c_2 - \phi_2 y_{t-1}\right)^2}{2\sigma^2}\right\} \end{bmatrix} .
\end{aligned}
\tag{2.4}
$$

Misalkan $\hat{\xi}_{t|t-1}$ melambangkan vektor (2×1) di mana elemen ke-*j* pada vektor merepresentasikan $P\{X_t = j|\mathrm{Y}_{t-1};\theta\}$ dan $\otimes$ melambangkan perkalian elemen per elemen vektor, maka

$$\begin{aligned}
\hat{\xi}_{t|t-1}\otimes\eta_t &= \begin{bmatrix} P\left(X_t = 1|\mathrm{Y}_{t-1};\theta\right) \\ P\left(X_t = 2|\mathrm{Y}_{t-1};\theta\right)\end{bmatrix} \otimes \begin{bmatrix} f\left(y_t|X_t = 1, \mathrm{Y}_{t-1};\theta\right) \\ f\left(y_t|X_t = 1, \mathrm{Y}_{t-1};\theta\right)\end{bmatrix} \\
&= \begin{bmatrix} P\{X_t = 1|\mathrm{Y}_{t-1};\theta\}\cdot f\left(y_t|X_t = 1, \mathrm{Y}_{t-1};\theta\right) \\ P\{X_t = 2|\mathrm{Y}_{t-1};\theta\}\cdot f\left(y_t|X_t = 2, \mathrm{Y}_{t-1};\theta\right)\end{bmatrix} \\
&= \begin{bmatrix} P\left(X_t = 1|\mathrm{Y}_{t-1};\theta\right)\cdot f\left(y_t|X_t = 1, \mathrm{Y}_{t-1};\theta\right) \\ P\left(X_t = 2|\mathrm{Y}_{t-1};\theta\right)\cdot f\left(y_t|X_t = 1, \mathrm{Y}_{t-1};\theta\right)\end{bmatrix}.
\end{aligned} \tag{2.5}$$

Berdasarkan persamaan (2.5) maka dapat dituliskan:

$$\begin{aligned}
&P\left(X_t = j|\mathrm{Y}_{t-1};\theta\right)\cdot f\left(y_t|X_t = j, \mathrm{Y}_{t-1};\theta\right) \\
&\qquad = \frac{P\left(X_t = j, \mathrm{Y}_{t-1};\theta\right)}{P\left(\mathrm{Y}_{t-1};\theta\right)}\cdot\frac{P\left(y_t, X_t = j, \mathrm{Y}_{t-1};\theta\right)}{P\left(X_t = j, \mathrm{Y}_{t-1};\theta\right)} \\
&\qquad = \frac{P\left(y_t, X_t = j, \mathrm{Y}_{t-1};\theta\right)}{P\left(\mathrm{Y}_{t-1};\theta\right)} = P\left(y_t, X_t = j|\mathrm{Y}_{t-1};\theta\right)
\end{aligned} \tag{2.6}$$

sehingga diperoleh

$$\begin{aligned}
f\left(y_t|\mathrm{Y}_{t-1};\theta\right) &= \sum_{j=1}^{2} P\left(y_t, X_t = j|\mathrm{Y}_{t-1};\theta\right) \\
&= \sum_{j=1}^{2} P\left(X_t = j|\mathrm{Y}_{t-1};\theta\right)\cdot f\left(y_t|X_t = j, \mathrm{Y}_{t-1};\theta\right) \\
&= P\left(X_t = 1|\mathrm{Y}_{t-1};\theta\right) f\left(y_t|X_t = 1, \mathrm{Y}_{t-1};\theta\right) \\
&\quad + P\left(X_t = 2|\mathrm{Y}_{t-1};\theta\right) f\left(y_t|X_t = 2, \mathrm{Y}_{t-1};\theta\right) \\
&= \mathbf{1}'\left(\hat{\xi}_{t|t-1}\otimes\eta_t\right),
\end{aligned} \tag{2.7}$$

di mana $\mathbf{1}' = [1 \quad 1]$.

Jika persamaan (2.6) dibagi dengan persamaan (2.7) maka diperoleh

$$\frac{P\left(y_t, X_t = j \middle| \mathrm{Y}_{t-1};\theta\right)}{f\left(y_t \middle| \mathrm{Y}_{t-1};\theta\right)} = \frac{P\left(y_t, X_t = j, \mathrm{Y}_{t-1};\theta\right)}{P\left(\mathrm{Y}_{t-1};\theta\right)} \cdot \frac{P\left(\mathrm{Y}_{t-1};\theta\right)}{P\left(y_t, \mathrm{Y}_{t-1};\theta\right)}$$
$$= \frac{P\left(y_t, X_t = j, \mathrm{Y}_{t-1};\theta\right)}{P\left(y_t, \mathrm{Y}_{t-1};\theta\right)} = \frac{P\left(X_t = j, y_t, \mathrm{Y}_{t-1};\theta\right)}{P\left(y_t, \mathrm{Y}_{t-1};\theta\right)}$$
$$= P\left(X_t = j \middle| y_t, \mathrm{Y}_{t-1};\theta\right) = P\left(X_t = j \middle| \mathrm{Y}_t;\theta\right). \tag{2.8}$$

Sehingga berdasarkan persamaan (2.6), (2.7) dan (2.8) diperoleh

$$P\left(X_t = j \middle| \mathrm{Y}_t;\theta\right) = \frac{P\left(y_t, X_t = j \middle| \mathrm{Y}_{t-1};\theta\right)}{f\left(y_t \middle| \mathrm{Y}_{t-1};\theta\right)}$$
$$\hat{\xi}_{t|t} = \frac{\left(\hat{\xi}_{t|t-1} \otimes \eta_t\right)}{\mathbf{1}'\left(\hat{\xi}_{t|t-1} \otimes \eta_t\right)}. \tag{2.9}$$
$$\xi_{t+1|t}^{(i)} = P\left(X_{t+1} = i \middle| \mathrm{Y}_t;\theta\right)$$
$$= \sum_{j=1}^{2} P\left(X_{t+1} = i, X_t = j \middle| \mathrm{Y}_t;\theta\right)$$
$$= \sum_{j=1}^{2} P\left(X_{t+1} = i \middle| X_t = j, \mathrm{Y}_t;\theta\right) \cdot P\left(X_t = j \middle| \mathrm{Y}_t;\theta\right)$$
$$= \sum_{j=1}^{2} P\left(X_{t+1} = i \middle| S_t = j;\theta\right) \cdot P\left(X_t = j \middle| \mathrm{Y}_t;\theta\right)$$
$$= \sum_{j=1}^{2} p_{ij} \xi_{t|t}^{(i)}$$

dan

$$\xi_{t+1|t} = \begin{bmatrix} p_{11} \cdot \xi_{t|t}^{(1)} + p_{12} \cdot \xi_{t|t}^{(2)} \\ p_{21} \cdot \xi_{t|t}^{(1)} + p_{22} \cdot \xi_{t|t}^{(2)} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} \xi_{t|t}^{(1)} \\ \xi_{t|t}^{(2)} \end{bmatrix} = \mathbf{P} \cdot \xi_{t|t}. \tag{2.10}$$

Fungsi log *likelihood* $\mathrm{L}\left(\theta\right)$ untuk data yang diamati Y_t dapat dihitung dengan cara

$$\mathrm{L}\left(\theta\right) = \log\left(f\left(y_1 \middle| \mathrm{Y}_0;\theta\right) f\left(y_2 \middle| \mathrm{Y}_1;\theta\right) \cdots f\left(y_T \middle| \mathrm{Y}_{T-1};\theta\right)\right) = \sum_{t=1}^{T} \log f\left(y_t \middle| \mathrm{Y}_{t-1};\theta\right) \cdot \tag{2.11}$$

Salah satu pendekatan yang dapat digunakan untuk memilih nilai awal bagi $\hat{\xi}_{t|t-1}$ adalah dengan membuat $\hat{\xi}_{1|0}$ sama dengan vektor peluang tak bersyarat $\pi = \begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix}^T$ yang memenuhi sifat *ergodic*, yaitu

$$\mathbf{P}\pi = \pi$$

$$\pi_1 + \pi_2 = 1$$

Sehingga diperoleh:

$$\hat{\xi}_{1|0} = \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} = \begin{bmatrix} P\left(X_t = 1 \middle| \mathrm{Y}_0 ; \theta\right) \\ P\left(X_t = 2 \middle| \mathrm{Y}_0 ; \theta\right) \end{bmatrix} = \begin{bmatrix} \dfrac{1 - p_{22}}{2 - p_{11} - p_{22}} \\ \dfrac{1 - p_{11}}{2 - p_{11} - p_{22}} \end{bmatrix}.$$

Berdasarkan persamaan (2.7) diperoleh

$$\begin{aligned} f\left(y_t \middle| \mathrm{Y}_{t-1};\theta\right) &= P\left(X_t = 1 \middle| \mathrm{Y}_{t-1};\theta\right) \cdot f\left(y_t \middle| X_t = 1, \mathrm{Y}_{t-1};\theta\right) \\ &\quad + P\left(X_t = 2 \middle| \mathrm{Y}_{t-1};\theta\right) \cdot f\left(y_t \middle| X_t = 2, \mathrm{Y}_{t-1};\theta\right) \\ &= P\left(X_t = 1 \middle| \mathrm{Y}_{t-1};\theta\right) \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ \frac{-\left(x_t - c_1 - \phi_1 y_{t-1}\right)^2}{2\sigma^2} \right\} \\ &\quad + P\left(X_t = 2 \middle| \mathrm{Y}_{t-1};\theta\right) \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ \frac{-\left(x_t - c_2 - \phi_2 y_{t-1}\right)^2}{2\sigma^2} \right\}. \end{aligned} \tag{2.12}$$

3. PENDUGAAN PARAMETER

Penduga kemungkinan maksimum bagi θ diperoleh dengan memaksimum-kan fungsi *log-likelihood*

$$\mathcal{L}\left(\theta\right) = \sum_{t=1}^{T} \log f\left(y_t \mid \mathrm{Y}_{t-1};\theta\right).$$

Berdasarkan persaman (2.12) diperoleh

$$\begin{aligned} \frac{\partial f\left(y_t \middle| \mathrm{Y}_{t-1};\theta\right)}{\partial c_1} &= P\left(X_t = 1 \middle| \mathrm{Y}_{t-1};\theta\right) \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ \frac{-\left(y_t - c_1 - \phi_1 y_{t-1}\right)^2}{2\sigma^2} \right\} \cdot \frac{\left(y_t - c_1 - \phi_1 y_{t-1}\right)}{\sigma^2} \\ &= P\left(X_t = 1 \middle| \mathrm{Y}_{t-1};\theta\right) f\left(y_t \middle| X_t = 1, \mathrm{Y}_{t-1};\theta\right) \frac{\left(y_t - c_1 - \phi_1 y_{t-1}\right)}{\sigma^2}. \end{aligned}$$

Untuk memperoleh nilai $\hat{c}_1$ yang memaksimum fungsi *log-likelihood* maka turunan pertama dari $\mathrm{L}\left(\theta\right)$ terhadap c_1 harus sama dengan nol.

$$\frac{\partial \mathcal{L}(\theta)}{\partial c_1} = 0$$

$$\Leftrightarrow \sum_{t=1}^{T} \frac{1}{f\left(y_t \middle| \Upsilon_{t-1};\theta\right)} \cdot \frac{\partial f\left(y_t \middle| \mathrm{Y}_{t-1};\theta\right)}{\partial c_1} = 0$$

$$\Leftrightarrow \sum_{t=1}^{T} \left[\frac{P\left(X_t = 1 \middle| \mathrm{Y}_{t-1};\theta\right) \cdot f\left(y_t \middle| X_t = 1, \mathrm{Y}_{t-1};\theta\right)}{f\left(y_t \middle| \mathrm{Y}_{t-1};\theta\right)} \cdot \frac{\left(y_t - c_1 - \phi_1 y_{t-1}\right)}{\sigma^2} \right] = 0$$

$$\Leftrightarrow \sum_{t=1}^{T} \frac{P\left(X_t = 1 \middle| \mathrm{Y}_{t-1};\theta\right) \cdot f\left(y_t \middle| X_t = 1, \mathrm{Y}_{t-1};\theta\right)}{f\left(y_t \middle| \mathrm{Y}_{t-1};\theta\right)} \cdot \left(y_t - \phi_1 y_{t-1}\right) =$$

$$c_1 \sum_{t=1}^{T} \frac{P\left(X_t = 1 \middle| \mathrm{Y}_{t-1};\theta\right) \cdot f\left(y_t \middle| X_t = 1, \mathrm{Y}_{t-1};\theta\right)}{f\left(y_t \middle| \mathrm{Y}_{t-1};\theta\right)}.$$

Sehingga diperoleh

$$\hat{c}_1 = \frac{\displaystyle\sum_{t=1}^{T} \frac{P\left(X_t = 1 \middle| \mathrm{Y}_{t-1};\hat{\theta}\right) \cdot f\left(y_t \middle| X_t = 1, \mathrm{Y}_{t-1};\hat{\theta}\right) \cdot \left(y_t - \hat{\phi}_1 y_{t-1}\right)}{f\left(y_t \middle| \mathrm{Y}_{t-1};\hat{\theta}\right)}}{\displaystyle\sum_{t=1}^{T} \frac{P\left(X_t = 1 \middle| \mathrm{Y}_{t-1};\hat{\theta}\right) \cdot f\left(y_t \middle| X_t = 1, \mathrm{Y}_{t-1};\hat{\theta}\right)}{f\left(y_t \middle| \mathrm{Y}_{t-1};\hat{\theta}\right)}} . \tag{3.1}$$

Dengan cara yang sama diperoleh:

$$\hat{c}_2 = \frac{\displaystyle\sum_{t=1}^{T} \frac{P\left(X_t = 2 \middle| \mathrm{Y}_{t-1};\hat{\theta}\right) \cdot f\left(y_t \middle| X_t = 2, \mathrm{Y}_{t-1};\hat{\theta}\right) \cdot \left(y_t - \hat{\phi}_2 y_{t-1}\right)}{f\left(y_t \middle| \mathrm{Y}_{t-1};\hat{\theta}\right)}}{\displaystyle\sum_{t=1}^{T} \frac{P\left(X_t = 2 \middle| \mathrm{Y}_{t-1};\hat{\theta}\right) \cdot f\left(y_t \middle| X_t = 2, \mathrm{Y}_{t-1};\hat{\theta}\right)}{f\left(y_t \middle| \mathrm{Y}_{t-1};\hat{\theta}\right)}} . \tag{3.2}$$

Berdasarkan persaman (2.12) diperoleh

$$\frac{\partial f\left(y_t \middle| \mathrm{Y}_{t-1};\theta\right)}{\partial \phi_1} = P\left(X_t = 1 \middle| \mathrm{Y}_{t-1};\theta\right) \cdot \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ \frac{-\left(y_t - c_1 - \phi_1 y_{t-1}\right)^2}{2\sigma^2} \right\} \cdot \frac{\left(y_t - c_1 - \phi_1 y_{t-1}\right) \quad \left(y_{t-1}\right)}{\sigma^2}$$

$$= P\left(X_t = 1 \middle| \mathrm{Y}_{t-1};\theta\right) \cdot f\left(y_t \middle| X_t = 1, \mathrm{Y}_{t-1};\theta\right) \cdot \frac{\left(y_t - c_1 - \phi_1 y_{t-1}\right) \quad \left(y_{t-1}\right)}{\sigma^2}.$$

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Untuk memperoleh nilai $\hat{\phi}_1$ yang memaksimum fungsi *log-likelihood* maka turunan pertama dari $\mathcal{L}(\theta)$ terhadap ϕ_1 harus sama dengan nol.

$$\frac{\partial \mathcal{L}(\theta)}{\partial \phi_1} = \sum_{t=1}^{T} \frac{1}{f\left(y_t \middle| \mathrm{Y}_{t-1};\theta\right)} \cdot \frac{\partial f\left(y_t \middle| \mathrm{Y}_{t-1};\theta\right)}{\partial \phi_1} = 0$$

$$\Leftrightarrow \sum_{t=1}^{T} \left[\frac{P\left(X_t = 1 \middle| \mathrm{Y}_{t-1};\theta\right) \cdot f\left(y_t \middle| X_t = 1, \mathrm{Y}_{t-1};\theta\right)}{f\left(y_t \middle| \mathrm{Y}_{t-1};\theta\right)} \cdot \frac{\left(y_t - c_1 - \phi_1 y_{t-1}\right)\ \left(y_{t-1}\right)}{\sigma^2} \right] = 0$$

memberikan

$$\sum_{t=1}^{T} \frac{P\left(X_t = 1 \middle| \mathrm{Y}_{t-1};\theta\right) f\left(y_t \middle| X_t = 1, \mathrm{Y}_{t-1};\theta\right)\left(y_{t-1}\right)\left(y_t - c_1\right)}{f\left(y_t \middle| \mathrm{Y}_{t-1};\theta\right)} =$$

$$\phi_1 \sum_{t=1}^{T} \frac{P\left(X_t = 1 \middle| \mathrm{Y}_{t-1};\theta\right) f\left(y_t \middle| X_t = 1, \mathrm{Y}_{t-1};\theta\right)\left(y_{t-1}\right)^2}{f\left(y_t \middle| \mathrm{Y}_{t-1};\theta\right)}.$$

Sehingga diperoleh

$$\hat{\phi}_1 = \frac{\displaystyle\sum_{t=1}^{T} \frac{P\left(X_t = 1 \middle| \mathrm{Y}_{t-1};\hat{\theta}\right) f\left(y_t \middle| X_t = 1, \mathrm{Y}_{t-1};\hat{\theta}\right)\left(y_{t-1}\right)\left(y_t - \hat{c}_1\right)}{f\left(y_t \middle| \mathrm{Y}_{t-1};\hat{\theta}\right)}}{\displaystyle\sum_{t=1}^{T} \frac{P\left(X_t = 1 \middle| \mathrm{Y}_{t-1};\hat{\theta}\right) f\left(y_t \middle| X_t = 1, \mathrm{Y}_{t-1};\hat{\theta}\right)\left(y_{t-1}\right)^2}{f\left(y_t \middle| \mathrm{Y}_{t-1};\hat{\theta}\right)}}. \tag{3.3}$$

Dengan cara yang sama diperoleh

$$\hat{\phi}_2 = \frac{\displaystyle\sum_{t=1}^{T} \frac{P\left(X_t = 2 \middle| \mathrm{Y}_{t-1};\hat{\theta}\right) f\left(y_t \middle| X_t = 2, \mathrm{Y}_{t-1};\hat{\theta}\right)\left(y_{t-1}\right)\left(y_t - \hat{c}_2\right)}{f\left(y_t \middle| \mathrm{Y}_{t-1};\hat{\theta}\right)}}{\displaystyle\sum_{t=1}^{T} \frac{P\left(X_t = 2 \middle| \mathrm{Y}_{t-1};\hat{\theta}\right) f\left(y_t \middle| X_t = 2, \mathrm{Y}_{t-1};\hat{\theta}\right)\left(y_{t-1}\right)^2}{f\left(y_t \middle| \mathrm{Y}_{t-1};\hat{\theta}\right)}}. \tag{3.4}$$

Berdasarkan persaman (2.12) diperoleh

$$\frac{\partial f\left(y_t \middle| \mathrm{Y}_{t-1};\theta\right)}{\partial \sigma^2} = \sum_{j=1}^{2}\left[-P\left(X_t = j \middle| \mathrm{Y}_{t-1};\theta\right)\frac{1}{2\sigma^3\sqrt{2\pi}}\exp\left\{\frac{-\left(y_t - c_j - \phi_j y_{t-1}\right)^2}{2\sigma^2}\right\}\right.$$

$$\left. + \left(P\left(X_t = j \middle| \mathrm{Y}_{t-1};\theta\right)\frac{1}{\sigma\sqrt{2\pi}}\exp\left\{\frac{-\left(y_t - c_j - \phi_j y_{t-1}\right)^2}{2\sigma^2}\right\}\cdot\frac{\left(y_t - c_j - \phi_j y_{t-1}\right)^2}{2\sigma^4}\right)\right]$$

$$= \sum_{j=1}^{2}\left[P\left(X_t = j \middle| \mathrm{Y}_{t-1};\theta\right)\frac{1}{\sigma\sqrt{2\pi}}\exp\left\{\frac{-\left(y_t - c_j - \phi_j y_{t-1}\right)^2}{2\sigma^2}\right\}\cdot\left(\frac{-1}{2\sigma^2} + \frac{\left(y_t - c_j - \phi_j y_{t-1}\right)^2}{2\sigma^4}\right)\right]$$

$$= \sum_{j=1}^{2}\left[P\left(X_t = j \middle| \mathrm{Y}_{t-1};\theta\right) f\left(y_t \middle| X_t = j, \mathrm{Y}_{t-1};\theta\right)\cdot\left(\frac{-1}{2\sigma^2} + \frac{\left(y_t - c_j - \phi_j y_{t-1}\right)^2}{2\sigma^4}\right)\right]$$

$$= \sum_{j=1}^{2}\left[P\left(X_t = j \middle| \mathrm{Y}_{t-1};\theta\right) f\left(y_t \middle| X_t = j, \mathrm{Y}_{t-1};\theta\right) \cdot\left(\frac{-\sigma^2}{2\sigma^4} + \frac{\left(y_t - c_j - \phi_j y_{t-1}\right)^2}{2\sigma^4}\right)\right].$$

Untuk memperoleh nilai $\hat{\sigma}^2$ yang memaksimum fungsi *log-likelihood* maka turunan pertama dari $\mathcal{L}(\theta)$ terhadap σ^2 harus sama dengan nol

$$\frac{\partial \mathcal{L}(\theta)}{\partial \sigma^2} = 0$$

$$\Leftrightarrow \sum_{t=1}^{T}\frac{1}{f\left(y_t \middle| \mathrm{Y}_{t-1};\theta\right)}\cdot\frac{\partial f\left(y_t \middle| \mathrm{Y}_{t-1};\theta\right)}{\partial \sigma^2} = 0$$

$$\Leftrightarrow \sum_{t=1}^{T}\sum_{j=1}^{2}\left[\frac{P\left(X_t = j \middle| \mathrm{Y}_{t-1};\theta\right) f\left(y_t \middle| X_t = j, \mathrm{Y}_{t-1};\theta\right)}{f\left(y_t \middle| \mathrm{Y}_{t-1};\theta\right)}\cdot\left(-\sigma^2 + \left(x_t - c_{S_t} - \phi_{S_t} y_{t-1}\right)^2\right)\right] = 0$$

memberikan

$$\sigma^2\sum_{t=1}^{T}\sum_{j=1}^{2}\frac{P\left(X_t = j \middle| \mathrm{Y}_{t-1};\theta\right) f\left(y_t \middle| X_t = j, \mathrm{Y}_{t-1};\theta\right)}{f\left(y_t \middle| \mathrm{Y}_{t-1};\theta\right)}$$

$$= \sum_{t=1}^{T}\sum_{j=1}^{2}\frac{P\left(X_t = j \middle| \mathrm{Y}_{t-1};\theta\right) f\left(y_t \middle| X_t = j, \mathrm{Y}_{t-1};\theta\right)\left(y_t - c_{S_t} - \phi_{S_t} y_{t-1}\right)^2}{f\left(y_t \middle| \mathrm{Y}_{t-1};\theta\right)}.$$

Sehingga diperoleh

$$\hat{\sigma}^2 = \frac{\sum_{t=1}^{T}\sum_{j=1}^{2}\frac{P\left(X_t = j \middle| \mathrm{Y}_{t-1};\hat{\theta}\right) f\left(y_t \middle| X_t = j, \mathrm{Y}_{t-1};\hat{\theta}\right)\left(y_t - \hat{c}_{S_t} - \hat{\phi}_{S_t} y_{t-1}\right)^2}{f\left(y_t \middle| \mathrm{Y}_{t-1};\hat{\theta}\right)}}{\sum_{t=1}^{T}\sum_{j=1}^{2}\frac{P\left(X_t = j \middle| \mathrm{Y}_{t-1};\hat{\theta}\right) f\left(y_t \middle| X_t = j, \mathrm{Y}_{t-1};\hat{\theta}\right)}{f\left(y_t \middle| \mathrm{Y}_{t-1};\hat{\theta}\right)}}. \qquad (3.5)$$

Untuk menduga parameter P digunakan formula (3.6) berikut yang diperoleh dari Hamilton (1990).

$$\hat{p}_{ij} = \frac{\sum_{t=2}^{T} P\left(X_t = j, X_{t-1} = i \mid \mathrm{Y}_T ; \hat{\theta}\right)}{\sum_{t=2}^{T} P\left(X_{t-1} = i \mid \mathrm{Y}_T ; \hat{\theta}\right)} \tag{3.6}$$

di mana menurut Kim (1994)

$$\begin{aligned}
&P\left(X_t = j, X_{t-1} = i \mid \mathrm{Y}_T ; \hat{\theta}\right) \\
&\quad = P\left(X_t = j \mid \mathrm{Y}_T ; \hat{\theta}\right) \times P\left(X_{t\text{-}1} = i \mid X_t = j, \mathrm{Y}_T ; \hat{\theta}\right) \\
&\quad \approx P\left(X_t = j \mid \mathrm{Y}_T ; \hat{\theta}\right) \times P\left(X_{t\text{-}1} = i \mid X_t = j, \mathrm{Y}_T ; \hat{\theta}\right) \\
&\quad = \frac{P\left(X_t = j \mid \mathrm{Y}_T ; \hat{\theta}\right) \times P\left(X_{t\text{-}1} = i, s_t = j \mid \mathrm{Y}_T ; \hat{\theta}\right)}{P\left(X_t = j \mid \mathrm{Y}_T ; \hat{\theta}\right)} \\
&\quad = \frac{P\left(X_t = j \mid \mathrm{Y}_T ; \hat{\theta}\right) \times P\left(X_{t\text{-}1} = i \mid \mathrm{Y}_T ; \hat{\theta}\right) \times P\left(X_t = i \mid X_{t\text{-}1} = i\right)}{P\left(X_t = j \mid \mathrm{Y}_T ; \hat{\theta}\right)}
\end{aligned} \tag{3.7}$$

dan

$$P\left(X_{t-1} = i \mid \mathrm{Y}_T ; \hat{\theta}\right) = \sum_{j=1}^{2} P\left(X_t = j, X_{t-1} = i \mid \mathrm{Y}_T ; \hat{\theta}\right).$$

4. PENDUGAAN ULANG PARAMETER DAN ALGORITMA

Karena persamaan (3.1) sampai (3.7) tak linear, maka tidak mungkin untuk memperoleh $\hat{\theta}$ secara analitik sebagai fungsi dari $\{Y_1, Y_2, Y_3, \cdots, Y_T\}$. Sehingga untuk mencari penduga kemungkinan maksimum digunakan algoritma iteratif dari metode EM.

4.1 Metode *Expectation Maximization* (Metode EM)

Algoritma EM dikembangkan oleh Baum and Petrie (1966) dengan ide dasar sebagai berikut.

Misalkan $\{P_\theta : \theta \in \Theta\}$ adalah koleksi ukuran peluang yang terdefinisi pada ruang (Ω, G) dan kontinu absolut terhadap P_0. Misalkan $\mathrm{Y} \subset \mathrm{G}$. Definisikan fungsi *likelihood* untuk menentukan penduga parameter θ berdasarkan informasi Y sebagai

$$L(\theta) = E_0\left[\frac{dP_\theta}{dP_0}\middle|\mathrm{Y}\right]$$

dan penduga maksimum *likelihood* didefinisikan sebagai

$$\hat{\theta} \in \arg\max_{\theta\in\Theta} L(\theta)\,.$$

Secara umum penduga maksimum likelihood $\hat{\theta}$ sulit dihitung secara langsung. Algoritma EM memberikan suatu metode iteratif untuk mengaproksimasi $\hat{\theta}$, dengan prosedur sebagai berikut.

Langkah 1: Set $p = 0$ dan pilih $\hat{\theta}_0$.

Langkah 2: [Langkah-E]

Set $\theta^* = \hat{\theta}_p$ dan hitung $Q\left(\theta,\theta^*\right) = E_{\theta^*}\left[\log\frac{dP_\theta}{dP_{\theta^*}}\middle|\mathrm{Y}\right]$.

Langkah 3: [Langkah-M]

Tentukan $\hat{\theta}_{p+1} \in \arg\max_{\theta\in\Theta} Q\left(\theta,\theta^*\right)$.

Langkah 4: $p \leftarrow p+1$

Ulangi langkah 2 sampai kriteria berhenti dipenuhi.

Catatan:

1. Barisan $\left\{\hat{\theta}_p : p \ge 0\right\}$ memberikan barisan $\left\{L\left(\hat{\theta}_p\right) : p \ge 0\right\}$ yang tak turun.
2. Menurut ketaksamaan Jensen,
$$Q\left(\hat{\theta}_{p+1},\hat{\theta}_p\right) \le \log L(\hat{\theta}_{p+1}) - \log L(\hat{\theta}_p)\,.$$
3. $Q\left(\theta,\theta^*\right)$ disebut *pseudo-loglikelihood* bersyarat.

4.2 Pendugaan ulang parameter menggunakan metode EM

Dengan menggunakan meode EM diperoleh algoritma untuk menduga ulang parameter model. Algoritma tersebut sebagai berikut.

Langkah 1:

Tentukan banyaknya data $\left(T\right)$ yang akan diamati serta tentukan juga nilai $\left(y_0, y_1, y_2, \cdots, y_{T-1}\right)$ dan matriks transisi

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}.$$

Beri nilai awal bagi $\hat{\theta}$ yang dilambangkan dengan $\hat{\theta}^{(m)} = \left(\hat{c}_1, \hat{c}_2, \hat{\phi}_1, \hat{\phi}_2, \hat{\sigma}^2\right)$.

Langkah 2:

Cari fungsi kerapatan bersyarat bagi y_T untuk setiap $t = 1, 2, \cdots, T$ dengan cara

$$\eta_t = \begin{bmatrix} f\left(y_t \middle| X_t = 1, \mathrm{Y}_{t-1}; \hat{\theta}^{(m)}\right) \\ f\left(y_t \middle| X_t = 2, \mathrm{Y}_{t-1}; \hat{\theta}^{(m)}\right) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\sqrt{2\pi}\hat{\sigma}} \exp\left\{ \dfrac{-\left(y_t - \hat{c}_1 - \hat{\phi}_1 y_{t-1}\right)^2}{2\hat{\sigma}^2} \right\} \\ \dfrac{1}{\sqrt{2\pi}\hat{\sigma}} \exp\left\{ \dfrac{-\left(y_t - \hat{c}_2 - \hat{\phi}_2 y_{t-1}\right)^2}{2\hat{\sigma}^2} \right\} \end{bmatrix}.$$

Langkah 3:

Penarikan kesimpulan optimal dan peramalan untuk setiap waktu *t* pada contoh dapat diperoleh melalui iterasi:

3.1. Tentukan nilai awal bagi $\hat{\xi}_{t|t-1}$ yang dilambangkan dengan $\hat{\xi}_{1|0}$.

3.2. Berikan nilai awal $i = 1$.

3.3. Untuk $t = i$, cari nilai dari

$$\begin{aligned} f\left(y_t \middle| \mathrm{Y}_{t-1}; \hat{\theta}^{(m)}\right) &= \mathbf{1}'\left(\hat{\xi}_{t|t-1} \otimes \eta_t\right) \\ &= P\left(X_t = 1 \middle| \mathrm{Y}_{t-1}; \hat{\theta}^{(m)}\right) \frac{1}{\sqrt{2\pi}\hat{\sigma}} \exp\left\{ \frac{-\left(y_t - \hat{c}_1 - \hat{\phi}_1 y_{t-1}\right)^2}{2\hat{\sigma}^2} \right\} \\ &\quad + P\left(X_t = 2 \middle| \mathrm{Y}_{t-1}; \hat{\theta}^{(m)}\right) \frac{1}{\sqrt{2\pi}\hat{\sigma}} \exp\left\{ \frac{-\left(y_t - \hat{c}_2 - \hat{\phi}_2 y_{t-1}\right)^2}{2\hat{\sigma}^2} \right\}, \end{aligned}$$

$$\hat{\xi}_{t|t} = \begin{bmatrix} P\left\{X_t = 1 \middle| \mathrm{Y}_{t-1}; \hat{\theta}^{(m)}\right\} \\ P\left\{X_t = 2 \middle| \mathrm{Y}_{t-1}; \hat{\theta}^{(m)}\right\} \end{bmatrix} = \frac{\left(\hat{\xi}_{t|t-1} \otimes \eta_t\right)}{\mathbf{1}'\left(\hat{\xi}_{t|t-1} \otimes \eta_t\right)},$$

$$\hat{\xi}_{t+1|t} = \begin{bmatrix} P\left\{X_{t+1} = 1 \middle| \mathrm{Y}_t; \hat{\theta}^{(m)}\right\} \\ P\left\{X_{t+1} = 2 \middle| \mathrm{Y}_t; \hat{\theta}^{(m)}\right\} \end{bmatrix} = \mathbf{P} \cdot \hat{\xi}_{t|t},$$

$i \leftarrow i + 1$.

3.4. Ulangi mulai dari langkah (3.3).
Stop jika $t = T$.
Lanjutkan ke langkah 4.

Langkah 4:

Cari nilai dari:

$$\hat{c}_i = \frac{\displaystyle\sum_{t=1}^{T} \frac{\hat{\xi}^{(j)}{}_{t+1|t} \cdot f\left(y_t \mid X_t = i, \mathrm{Y}_{t-1}; \hat{\theta}\right)}{f\left(y_t \mid \mathrm{Y}_{t-1}; \hat{\theta}\right)} \cdot \left(y_t - \phi_i y_{t-1}\right)}{\displaystyle\sum_{t=1}^{T} \frac{\hat{\xi}^{(j)}{}_{t+1|t} \cdot f\left(y_t \mid X_t = i, \mathrm{Y}_{t-1}; \hat{\theta}\right)}{f\left(y_t \mid \mathrm{Y}_{t-1}; \hat{\theta}\right)}}, \quad i = 1, 2.$$

$$\hat{\phi}_i = \frac{\displaystyle\sum_{t=1}^{T} \frac{\hat{\xi}^{(j)}{}_{t+1|t} \cdot f\left(y_t \mid X_t = i, \mathrm{Y}_{t-1}; \hat{\theta}\right)}{f\left(y_t \mid \mathrm{Y}_{t-1}; \hat{\theta}\right)} \cdot \left(y_t - \hat{c}_i\right)\left(y_{t-1}\right)}{\displaystyle\sum_{t=1}^{T} \frac{\hat{\xi}^{(j)}{}_{t+1|t} \cdot f\left(y_t \mid X_t = i, \mathrm{Y}_{t-1}; \hat{\theta}\right)\left(y_{t-1}\right)^2}{f\left(y_t \mid \mathrm{Y}_{t-1}; \hat{\theta}\right)}}, \quad i = 1, 2.$$

$$\hat{\sigma}^2 = \frac{\displaystyle\sum_{t=1}^{T}\sum_{j=1}^{2} \frac{\hat{\xi}^{(j)}{}_{t+1|t}\, f\left(y_t \mid X_t = j, \mathrm{Y}_{t-1}; \hat{\theta}\right)\left(y_t - \hat{c}_{X_t} - \hat{\phi}_{X_t} y_{t-1}\right)^2}{f\left(y_t \mid \mathrm{Y}_{t-1}; \theta\right)}}{\displaystyle\sum_{t=1}^{T}\sum_{j=1}^{2} \frac{\hat{\xi}^{(j)}{}_{t+1|t}\, f\left(y_t \mid X_t = j, \mathrm{Y}_{t-1}; \hat{\theta}\right)}{f\left(y_t \mid \mathrm{Y}_{t-1}; \hat{\theta}\right)}}.$$

<u>Langkah 5:</u>

Beri nama parameter yang dihasilkan pada langkah 4 dengan $\hat{\theta}_{k+1} = \left(\hat{c}_1, \hat{c}_2, \hat{\phi}_1, \hat{\phi}_2, \hat{\sigma}^2\right)$, $k = 0, 1, 2, \cdots, T-1$.

<u>Langkah 6:</u>

Cari **P** yang baru, yaitu:

$$\hat{\xi}^{(j)}_{t|T} = \hat{\xi}^{(j)}_{t|t} \otimes \left\{\mathbf{P}'\left[\hat{\xi}^{(j)}_{t+1|T} (\div) \hat{\xi}^{(j)}_{t+1|t}\right]\right\}$$

di mana $(\div)$ merupakan operasi perkalian elemen per elemen vektor,

$$\begin{aligned} P\left(X_t = j, X_{t-1} = i \mid \mathrm{Y}_T\right) &\approx \frac{P\left(X_t = j \mid \mathrm{Y}_T\right) \times P\left(X_{t-1} = i \mid \mathrm{Y}_t\right) \times P\left(X_t = i \mid X_{t-1} = i\right)}{P\left(X_t = j \mid \mathrm{Y}_t\right)} \\ &= \frac{\hat{\xi}^{(j)}{}_{t|T} \times \hat{\xi}^{(i)}{}_{t-1|t-1} \times p_{ij}}{\hat{\xi}^{(j)}{}_{t|t}}, \end{aligned}$$

$$P\left(X_{t-1}=i \mid \mathrm{Y}_t;\hat{\theta}\right)=\sum_{j=1}^{2} P\left(X_t = j, X_{t-1}=i \mid \mathrm{Y}_T\right),$$

$$\hat{p}_{ij} = \frac{\sum_{t=2}^{T} P\left(X_t = j, X_{t-1}=i \mid \mathrm{Y}_T;\hat{\theta}\right)}{\sum_{t=2}^{T} P\left(X_{t-1}=i \mid \mathrm{Y}_T;\hat{\theta}\right)} = \frac{\sum_{t=2}^{T} \dfrac{\hat{\xi}^{(j)}{}_{t|T} \times \hat{\xi}^{(i)}{}_{t-1|t-1} \times p_{ij}}{\hat{\xi}^{(j)}{}_{t|t}}}{\sum_{t=2}^{T}\sum_{j=1}^{2} \dfrac{\hat{\xi}^{(j)}{}_{t|T} \times \hat{\xi}^{(i)}{}_{t-1|t-1} \times p_{ij}}{\hat{\xi}^{(j)}{}_{t|t}}}.$$

Langkah 7:

Selama $k < T$, ulangi mulai dari langkah 2. Gunakan parameter yang sudah dihasilkan untuk mencari nilai harapan bagi nilai *y* yang akan datang.

$$E\left[Y_{t+1} \mid X_{t+1} = j, \mathrm{Y}_t;\theta\right] = c_j + \phi_j y_t.$$

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