MIXED FINITE ELEMENT FORMULATION OF THE BIHARMONIC EQUATION

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ABSTRACT. We will provide an abstract setting for mixed finite element method for biharmonic equation. The abstract setting casts mixed finite element method for first biharmonic equation and second biharmonic equation into a single framework altogether. We provide error estimates for both type biharmonic equation, and for the first time an error estimate for the second biharmonic equation.

In this note we will discuss the mixed finite element method for Biharmonic problem. The result of which depend heavily on the results of Brezzi and Raviart [2]. It should mentioned that the approach taken by them could accommodate the first and second Biharmonic problem nicely. They developed their abstract methods initially aimed at a unified frameworks of several mixed finite element method for plate equation into a single setting. The presentation of this note is arranged as follow. In the first section we will take a look at two variational formulations of the first Biharmonic problem. The first formulation is a direct variational formulation of the plate problem, and second is mixed variational formulation formulation due to Ciarle and Raviart [4], it can be shown that their formulation is a special case of Brezzi and Raviart abstract setting. In the second section, we present an abstract result due to Brezzi and Raviart [2]. This result is applied to the first and second Biharmonic problems in section 3 and 4 respectively, where also included some error estimate for both equations.

1. Variational Problem

1.1. **Direct Variational.** Let $f \in L^2(\Omega)$ be given, consider the problem (P_b) :
Find $u \in V = H_0^2$ which satisfies:

$$a(u,v) = f(v), \forall v \in V$$

where the bilinear form on V defined as :

$$a(u,v) = \int_\Omega \{\sigma \Delta u \Delta v + (1-\sigma)(\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial y^2} + 2(\frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 v}{\partial x \partial y}))\}, \quad \forall v \, inV,$$

and linear form f on V defined as:

$$f(v) = \int_\Omega fv, \quad \forall v \in V.$$

From discussion on thin plate deformation, this problem is related to minimization of quadratic functional energy:

$$J(v) = a(u,v) - f(v).$$

Since :

$$a(v,v) = \sigma|\Delta v|_0 + (1-\sigma)|v|_0, \forall v \in H_0^2(\Omega),$$

then obviously $a(v,v)$ is $V-$elliptic and continuous over V, hence problem (P_b) is uniquely solvable for every $f \in L^2(\Omega)$, by virtues of Lax-Milgram Lemma [Jo90]

If the solution $u \in H^4 \cap H_0^2$, i.e. smooth enough, then this is a solution of the plate equation :

$$\begin{aligned} \Delta^2 u &= f \\ u|_{\partial\Omega} = 0, \qquad & \frac{\partial u}{\partial n}|_{\partial\Omega} = 0 \end{aligned}$$

Now, we have to choose the finite element space $V_{h,0} \subset V = H_0^2$ within which we have to solve the following discrete problem:
Find $u_h \in V_{h,0}$ which satisfies:

$$a(u_h, v_h) = f(v_h), \forall v_h \in V_{h,0}.$$

The inclusion $V_{h,0} \subset V = H_0^2$ in turn implies that $V_{h,0} \subset C^1(\bar{\Omega})$ which in itself implies requirement of the degree of freedom to generate basis function. For example, we may choose as a basis over triangles is Argyris basis which is a complete polynomial of order 5.

1.2. **Mixed Variational.** Another approach to solve the first biharmonic equation was proposed by Ciarlet and Raviart [CR74] in the following way. Introduce a new variable $\psi = \Delta v$. Then the minimization of energy functional of plate equation becomes:

$$\begin{aligned} \min J(v) &= \min\{\int_\Omega (|\psi|^2 - f\cdot v)\}, \\ \text{subject to :} \quad & \psi = \Delta v. \end{aligned}$$

In order to solve this formulation, let us introduce the following space :
$X = H_0^1 \times L^2$. Define on X the bilinear form: $\tilde{a} : X \times X \mapsto R$ by

$$\tilde{a}(x,y) = \int_\Omega \phi\,\psi, \quad \forall (x,y) = ((u,phi),(v,\psi)) \in X \times X.$$

This bilinear form is obviously symmetric and continuous by introducing on X the product norms:

$$\|x\|_X = |u|_1 + |\phi|_0,\ x = (u,\phi).$$

Likewise, the linear form $f(-)$ defined on X by:

$$f(y) = \int_\Omega f\psi, y = (v,\psi)$$

is continuous.

Next, let $M = H^1$, defined on $X \times M$ a bilinear form by:

$$b(y,\mu) = \int_\Omega \nabla u \nabla \mu - \int_\Omega \psi\mu,\ \forall((v,\psi),\mu) \in X \times M,$$

which is clearly continuous. Finally, let $\tilde{X} = \{x = (v,\psi) \in X;\ \forall \mu \in M, b(x,\mu) = 0\}$. Clearly $\tilde{X}$ is a Hilbert space since it is a closed subspace of X. Moreover, if $y = (v,\psi) \in X$ satisfy; $b(y,\mu) = 0, \forall \mu \in M$, then we have in particular:

$$\int_\Omega \nabla u \nabla v\, dx = \int_\Omega \psi\mu\, dx, \quad \forall \mu \in H_0^1,$$

hence v appears as a solution for Laplace's equation with homogeneous Dirichlet conditions on Ω. Assuming Ω is a convex domain, implies $v \in H^2$ by regularity of solutions of second order problem. Using Green's formula :

$$\int_\Omega \nabla v \nabla \mu\, dx = -\int_\Omega \Delta v\ \mu\, dx + \int_{\partial\Omega} \partial_\nu v \mu.$$

We deduce that $-\Delta v = \psi$. Observe that since the product norms is equivalent to the norm defined by : $|v| : y \mapsto |\psi|_0, y = (v,\psi) \in \tilde{X}$, by Theorem 7.1.1. of Ciarlet [**?**], so we can show that the bilinear form $\tilde{a}$ is $V-$elliptic in the space $\tilde{X}$.

Theorem 1.1. *Let $u \in H_0^2(\Omega)$ denote the minimizer of energy functional of plates. Then we also have:*

$$J(u, -\Delta u) = \inf_{(u,\psi)\in\tilde{X}} J(u, \psi).$$

Additionally, the pair $(u, -\Delta u) \in \tilde{X}$ is the unique solution of the above minimization problem.

2. Abstract Result

Let V abd W be two (real) Hilbert spaces with norms $\| - \|_V$ and $\| - Vert_W$ respectively. Let V' and W' be the dual spaces of V and W respectively. Denote by $[.,.]$ the pairing between V' and V or W' and W.

Let $a(.,.)$ and $b(.,.)$ be two continuous bilinear forms on $V \times V$ and $V \times W$. Set:

(A.1) $\qquad \|a\| = \sup_{v\in V} \frac{|a(u,v)|}{\|u\|_V \|v\|_V}$

Assume that the bilinear form $a(.,.)$ is $V-$elliptic, in a sense that there exist a constant $\alpha > 0$ such that:

(A.1) $\qquad \forall v \in V, a(v,v) \geq \alpha \|v\|_V^2$

In addition assume that there exists a constant $\beta > 0$ such that:

(A.3) $\qquad \forall w \in W, \sup_{v\in V} \frac{|b(v,w)|}{\|v\|_V} \geq \beta \|w\|_W$

Consider the following problem (P): Given $f \in V'$ and $g \in W'$, find a pair $(u,p) \in V \times W$ which satisfies:

$$\forall v \in V, a(u,v) + b(v,p) = [f,v] \tag{2.1}$$
$$\forall w \in W, b(u,w) = [g,w] \tag{2.2}$$

we recall the following results due to Brezzi [3]:

Theorem 2.1. *Assume that the hypothesis (A.2) and (A.3) hold. Then problem (P) has unique solution $(u,p) \in V \times W$.*

See [6].

Next we shall give another characterization of the problem (P) well suited for mixed finite element methods. We are given two other (real) Hilbert spaces $\tilde{V}$ and $\tilde{W}$, with norms $\| - \|_{\tilde{V}}$ and $\| - \|_{\tilde{W}}$ respectively, such that $\tilde{V} \subset V$ and $W \subset \tilde{W}$ with continuous imbedding, and W is dense in $\tilde{W}$. Again let $[.,.]$ denote the pairing between $\tilde{W}$ and its dual $\tilde{W}'$. Assume that there exists a continuous bilinear form $\tilde{b}(.,.)$ on $\tilde{V} \times \tilde{W}$. With the following properties:

$$(A.4) \qquad \forall v \in V, \forall w \in \tilde{W}, \tilde{b}(v,w) = b(v,w) \forall w \in \tilde{W}$$

$$(A.5) \qquad \forall w \in \tilde{W}, \sup_{v\in\tilde{V}} \frac{|\tilde{b}(v,w)|}{\|v\|_{\tilde{V}}} \geq \tilde{\beta} \|w\|_W$$

for some constant $\tilde{\beta} > 0$. Set:

$$\|\tilde{b}\| = \sup_{v\in\tilde{V},w\in\tilde{W}} \frac{|\tilde{b}(v,w)|}{\|wVert_{\tilde{W}}\|v\|_{\tilde{V}}}.$$

Assume that $g \in \tilde{W}'$ and consider the following problem $(\tilde{P})$:
Find $(u,p) \in \tilde{V} \times \tilde{W}$ such that:

$$\forall v \in \tilde{V}, a(u,v) + \tilde{b}(v,p) = [f,v] \tag{2.3}$$

$$\forall w \in \tilde{W}, \tilde{b}(u,w) = [g,w] \tag{2.4}$$

Observe that $a(.,.)$ is not a-priori $\tilde{V}-$elliptic. However we have:

Theorem 2.2. *Assume conditions (A.4) and (A.5) hold and that the first argument u of the solution (u,p) of problem (P) belongs to $\tilde{V}$. Then (u,p) is the unique solution of problem $(\tilde{P})$.*

Please consult [6] for proof.

In the sequel, assume hat the conclusion of Theorem 2 holds. Let us now consider some general results of approximations of the solutions (u,p) of problem $(\tilde{P})$. Given V_h and W_h two finite dimensional spaces such that:

$$V_h \subset \tilde{V}, W_h \subset tildeW$$

Consider the following approximate problem $(\tilde{P}_h)$:
Find $(u_h,p_h) \in V_h \times W_h$ such that:

$$\forall v_h \in V_h, a(u_h,v_h) + \tilde{b}(v_h,p_h) = [f,v_h] \tag{2.5}$$

$$\forall w_h \in W_h, \tilde{b}(u,w) = [g,w_h] \tag{2.6}$$

As corollary of the previous theorem, we have the following.

Theorem 2.3. *Assume that*

$$\forall w_h \in \tilde{W}_h, \sup_{v_h\in\tilde{V}_h} \frac{|\tilde{b}(v_h,w_h)|}{\|v_h\|_{\tilde{V}_h}} \geq \gamma\|w_h\|_{\tilde{W}} \qquad \cdots (A.6)$$

Then problem (P_h) has unique solution $(u_h,p_h) \in V_h \times W_h$.

Now we will derive some abstract bounds on the errors $e_1 = (u - u_h)$ and $e_2 = (p - p_h)$. The approach is as follows. Since V_h is finite dimensional, there exists $S(h) > 0$, which depends on V_h such that :

$$\forall v_h \in V_h, \|v_h\|_{\tilde{V}} \leq S(h)\|v_h\|_V$$

Let us set:

$$Z = \{v \in \tilde{V}; \forall w \in W, \tilde{b}(v,w) = 0\},$$

and for any $\phi \in W'$,

$$Z_h = \{v_h \in V_h; \forall w_h \in W_h, \tilde{b}(v_h,w_h) = [\phi,w_h]\}.$$

It is clear that the subspace Z_h of V_h may be regarded as a finite dimensional approximation for Z, even though in general it is not contained in Z.

Theorem 2.4. *Assume hypothesis (A.2) and (A.6) hold. Then there exists a constant* $K = K(\alpha, \gamma, \|a\|, \|\tilde{b}\|)$ *such that:*

$$\|e_1\|_V + \|e_2\|_{\tilde{W}} \leq K\{\inf_{v_h \in V_h} \|e_1\|_{\tilde{V}} + (1 + S(h)) \inf_{w_h \in W_h} \|e_2\|_{W_h}\}$$

For the proof, see [6].

Brezzi and Raviart noted that the error bound in the above theorem is not optimal for the following reasons. The estimates involve the error of approximation $\inf_{V_h} \|u - v_h\|_{\tilde{V}}$ in $\tilde{V}$ instead of the corresponding error $\inf_{V_h} \|u - v_h\|_V$ in V. At the same time the constant $S(h)$ is never bounded independently of the finite dimensional space V_h but tends to infinity as $\dim(V_h)$ increases. By adding supplementary hypothesis, however, the error bound can be improved.

Theorem 2.5. *Assume the hypothesis in the previous theorem are holds. Additionally, assume the inclusion* $Z_h \subset Z$ *holds. Then there exist a constant* $K = K(\alpha, \gamma, \|a\|, \|\tilde{b}\|)$ *such that:*

$$\|e_1\|_V + \|e_2\|_{\tilde{W}} \leq K\{\inf_{v_h \in Z_h(\phi)} \|e_1\|_V + \inf_{w_h \in W_h} \|e_2\|_{\tilde{W}}\},$$

where $Z_h(\phi)$ *is defined as above.*

Instead of the above theorem, we shall use the following consequences.

Corollary 2.6. *Assume (A.1), (A.5), and inclusion* $Z_h \subset Z$ *hold. Assume that there exists an operator* $\pi_h : \tilde{V} \longrightarrow V_h$ *and a constant* $c > 0$ *such that*

$$\begin{aligned} \forall w_h \in W_h, \tilde{b}(v - \pi_h v, w_h) &= 0 \\ \forall v \in \tilde{V}, \|\pi_h v\|_{\tilde{V}} &\leq c\|v\|_{\tilde{V}}. \end{aligned}$$

Then the problem $(\tilde{P}_h)$ *has unique solution* $(u_h, p_h) \in V_h \times W_h$ *and we have:*

$$\|e_1\|_V + \|e_2\|_{\tilde{W}} \leq K\{\|u - \pi_h u\|_V + \inf_{w_h \in W_h} \|e_2\|_{\tilde{W}}\},$$

for a constant $K = K(\alpha, \tilde{\beta}, c, \|a\|, \|\tilde{b}\|)$.

For complete proof, see the thesisi of Garnadi [6].

Let Λ be a Hilbert space with inner product $< ., . >_\Lambda$ and its induced norms $\|.\|_\Lambda$ such that $\tilde{W} \subset \Lambda$ with continuous imbedding.

Assume that problem (P) is regular in the following sense.

Given $\psi \in \Lambda$, let $(y, \Xi) \in V \times \Lambda$ be the solutions of:

$$\forall v \in V, a(v, y) + \tilde{b}(v, \Xi) = 0 \tag{2.7}$$

$$\forall w \in \tilde{W}, \tilde{b}(y, w) = < \psi, w >_\Lambda . \tag{2.8}$$

Then y belongs to the space $\tilde{V}$.

Theorem 2.7. *Assume the problem (P) is regular as in the previous sense. Then we have a constant* $K = K(\|a\|, \|\tilde{b}\|) > 0$,

$$\begin{aligned} \|p - w_h\|_\Lambda \quad &\leq \quad \sup_{\psi \in \Lambda}\{\frac{1}{\|\psi\|} \\ & \inf_{y_h \in Z_h(\phi), w_h \in W_h, \Xi_h \in W_h} [\|e_1\|_V \|y - y_h\|_V + \|e_1\|_{\tilde{V}} \|\Xi - \Xi_h\|_V + \|e_2\|_{\tilde{W}}]\}, \end{aligned}$$

where, for any given $\psi \in \Lambda$, (y, Ξ) *is the regular solution in the previous sense.*

3. Application to First Biharmonic Equation

We consider the first biharmonic equation:

$$\begin{aligned} \Delta^2 u \quad &= \quad g, \text{in} \quad \Omega \\ u|_{\partial\Omega} \quad &= \quad \frac{\partial u}{\partial n}|_{\partial\Omega} = 0. \end{aligned}$$

Rewriting the problem as a coupled system as follow:

$$\begin{aligned} w - \Delta u \quad &= \quad 0, \text{in} \quad \Omega \\ \Delta w \quad &= \quad g, \text{in} \quad \Omega \\ u|_{\partial\Omega} \quad &= \quad \frac{\partial u}{\partial n}|_{\partial\Omega} = 0. \end{aligned}$$

Set:

$$\begin{array}{cc} V = L^2(\Omega) & W = H_0^2(\Omega) \\ a(u,v) = \int uv \, dx & b(v,w) = \int v\Delta \, dx \\ f = 0 & [\phi, \mu] = -\int \phi\mu \, dx. \end{array}$$

We will check whether the hypothesis are satisfied. Since $a(u,v)$ in V is an inner product in $L^2(\Omega)$, by taking $\alpha = 1$, then $a(u,v)$ is $V-$elliptic. Clearly that the first hypothesis is satisfied. And we can verify that there exist $\beta \geq 0$ such that

$$\forall w, W, \sup_{v \in V} \frac{b(v,w)}{\|v\|_V} \geq \beta \|w\|_W.$$

We associated the problem (B.2) with a problem ($\tilde{B}$.2) by setting

$$\tilde{V} = H^1(\Omega) \qquad ; \quad \tilde{W} = H_0^1(\Omega) \qquad ; \quad \tilde{b}(v,w) = -\int \nabla v \nabla w \, dx.$$

The conditions (A.1) and (A.2) hold trivially and by Theorem 2, there exists a pair of functions $(u,p) \in H^1(\Omega) \times H_0^1(\Omega)$ which is the unique solution of:

$$\begin{aligned} \int vu \, dx \; + \int \nabla v \nabla p \, dx \quad &= \quad 0, \forall v \in H^1(\Omega) \\ \int \nabla p \nabla u \, dx \; + \int gw \, dx \quad &= \quad 0, \forall w \in H_0^1(\Omega). \end{aligned}$$

Let us assume that the solution be an element in $H_0^2 \cap H^3$. Suppose that Ω is triangulated by triangulation T_h with triangles K whose sides are less than h. Define finite dimensional space :

$$V_h = \{v_h \in C^0(\Omega); K \in T_h, v_h|_K \in P_1(K)\}$$

$$V_{h,0} = V_h \cap H_0^1 \qquad W_h = V_{h,0}.$$

So we take the piecewise linear basis. In order to apply Theorem 3, we need to check the hypothesis (A.6).

Lemma 3.1. *Let there be given spaces V_h and W_h as above. Then there exist a constant $\gamma > 0$ independent ofh such that consequences in Theorem 3 holds.*

See [6] for proof.

Let us define Ritz operators $R_h : H^1 \to V_h$ and $R_{h,0} : H_0^1 \to V_{h,0}$ as follows

$$\begin{aligned} \tilde{b}(v - R_h v, \eta) &= 0, \forall \eta \in V_h \\ \int (v - R_h v)\, dx &= 0, \end{aligned}$$

and

$$\tilde{b}(v - R_{h,0} v, \xi) = 0, \forall \xi \in V_{h,0},$$

respecively.

The following lemma is important to the proof of estimates for piecewise linear basis. The proof uses $L^\infty-$error estimates for the Ritz approximation of second order problem.

Lemma 3.2. *For all $u \in H_0^1 \cap W^{2,\infty}$ and for all $\eta \in V_h$ we have*

$$\tilde{b}(v - R_{h,0} v, \eta) = ch^{1/2} |\ln h| \|\Delta u\|_\infty \|\eta\|_0$$

with c is independent of u, η, and h.

See [6] for proof.

Let $(u_h, p_h) \in V_{h,0} \times V_h$ be the solution of discrete problem. With $e_1 = (u - u_h)$ and $e_2 = (p - p_h)$, we can rewrite equations in $(\tilde{B}.1\ h)$ in the following form:

$$\begin{aligned} \tilde{b}(e_1, \eta) &= (e_2, \eta), \forall \eta \in V_h \\ \tilde{b}(e_2, \xi) &= 0, \forall \xi \in V_{h,0}. \end{aligned}$$

We obtain the following estimates in the L^2-norm.

Theorem 3.3. *The differences of e_1 and e_2 between the exact solution of the first biharmonic problem and the mixed finite element formulation can be estimated by :*

$$\|e_1\|_0 + h^{1/2} |\ln h|\ \|e_2\|_0 \le c\ h |\ln(h)|^2 \|u\|_4$$

where c is independent of (u, p) and h.

Consult [6] for proof.

Corollary 3.4. *As a consequence of estimates in the above theorem and combination of the first part of previous equation we get for* e_1*in the* H^2-*norm*

$$\|e_1\|_2 \le c\, h^{3/4}|\ln(h)|^{3/2}\|u\|_4.$$

Observe that Scholz result need a higher regularity, i.e. $u \in H^4 \cap H_0^1$ to show error estimates in L^2 for piecewise linear basis, while for approximation using $P_k, k \ge 2$, error estimates in $O(h^{k-1})$ is achieved, with weaker regularity, $u \in H^3 \cap H_0^1$ [Mo87]. Furthermore, Glowinski [Gl73] showed that for a particular triangulation of a square estimates in $O(h)$ is achieved.

4. Application to Second Biharmonic Equation

We consider the second biharmonic equation:

$$\begin{aligned} \Delta^2 u &= g, \text{in} \quad \Omega \\ u|_{\partial\Omega} &= \Delta u|_{\partial\Omega} = 0. \end{aligned}$$

Rewriting the problem as a coupled system as follow:

$$\begin{aligned} w - \Delta u &= 0, \text{in} \quad \Omega \\ \Delta w &= g, \text{in} \quad \Omega \\ u|_{\partial\Omega} &= w|_{\partial\Omega} = 0. \end{aligned}$$

Set:

$$\begin{array}{cc} V = L^2(\Omega) & W = H_0^2(\Omega) \\ a(u,v) = \int uv\, dx & b(v,w) = \int v\Delta\, dx \\ f = 0 & [\phi,\mu] = -\int \phi\mu\, dx. \end{array}$$

We need to check whether the hypothesis are satisfied. The arguments are follows verbatimly as in the case of the first biharmonic equation in the previous section, however as the space and functional forms are slightly different, it is necessary to follows the arguments step by step. Since $a(u,v)$ in V is an inner product in $L^2(\Omega)$, by taking $\alpha = 1$, then $a(u,b)$ is $V-$elliptic. It is clear that the first hypothesis is satisfied. And we can verify that there exist $\beta \ge 0$ such that

$$\forall w, W, \sup_{v\in V} \frac{b(v,w)}{\|v\|_V} \ge \beta\|w\|_W.$$

We associated the problem (B.2) with a problem ($\tilde{B}$.2) by set

$$\tilde{V} = H_0^1(\Omega) \qquad ; \quad \tilde{W} = H_0^1(\Omega) \qquad ; \quad \tilde{b}(v,w) = -\int \nabla v \nabla w\, dx.$$

The conditions (A.1) and (A.2) hold trivially and by Theorem 2, there exists a pair of functions $(u,h) \in H_0^1(\Omega) \times H_0^1(\Omega)$ which is the unique solution of:

$$\begin{aligned} \int vu \, dx \ + \int \nabla v \nabla p \, dx &= 0, \forall v \in H_0^1(\Omega) \\ \int \nabla p \nabla u \, dx \ + \int gw \, dx &= 0, \forall w \in H_0^1(\Omega). \end{aligned}$$

Let us assume that the solution be an element in $H_0^2 \cap H^3$. Suppose that Ω is triangulated by triangulation T_h with triangles K whose sides are less than h. Define finite dimensional space :

$$\begin{aligned} V_h &= \{v_h \in C^0(\Omega); K \in T_h, v_h|_K \in P_1(K)\} \\ V_{h,0} V_h \cap H_0^1 & \qquad W_h = V_{h,0}. \end{aligned}$$

So we take the piecewise linear basis. In order to apply Theorem 3, we need to check the hypothesis (A.6). This is obviously clear, since $H_0^1 \subset H^1$.

The error bounds for first biharmonic problems can be extended to the second biharmonic problems with only minor changes in the proof. Furthermore, error bounds for Poisson equation can be easily extended to the second biharmonic problems by treating the splitting of Biharmonic equation as a coupled set of Poisson equations. Then we have the following theorems as consequences.

Theorem 4.1. i. *If $u \in H^4(\Omega)$, then*

$$\begin{aligned} \|u - P_h u\|_0 &\leq c_1 h^2 |u|_2 \\ \|p - P_h p\|_0 &\leq c_1 h^2 |p|_2, \end{aligned}$$

Then,

$$\|u - P_h u\|_0 + \|p - P_h p\|_0 \leq h^2 (c_1 |u|_2 + c_2 |p|_2)$$

ii. *If $u \in H^4(\Omega)$, then*

$$\begin{aligned} \|u - R_h u\|_0 &\leq c_1 h^2 |u|_2 \\ \|p - R_h p\|_0 &\leq c_1 h^2 |p|_2, \end{aligned}$$

Then,

$$\|u - R_h u\|_0 + \|p - R_h p\|_0 \leq h^2 (c_1 |u|_2 + c_2 |p|_2)$$

Similarly, from L^2-estimates for Poisson equation, we have directly the following results on L^2-estimates of second biharmonic problems.

Theorem 4.2.

i.
$$\|u - R_h u\|_0 \leq c_1 h^2 |u|_2 \leq c_1 h^2 \|p\|_0$$

ii.
$$\|p - R_h p\|_0 \leq c_1 h^2 |p|_2 \leq \|f\|_0.$$

Note that both previous theorems is one of the results in this works, which to the author knowledge has not appeared elsewhere, albeit it is seemingly light exercises.

5. Conclusion

We demonstrate the abstract setting encompassed not only several mixed finite element methods for the first biharmonic equation, but also accommodate the second biharmonic equation. One important thing in this work is an error estimate for the second biharmonic equation which has not appeared elsewhere. Furthermore, the error estimates provided in this work will be of useful to extend of adaptivity for works on thin plate finite element interpolation for sparse data [12, 8, 9], which its variational form has a close resemblance with mixed finite element formulation of biharmonic equation. We are looking forward for extension of this work to thin plate finite element interpolation in the future.

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