HIGHER ORDER KORTEWEG-DE VRIES MODELS FOR INTERNAL WAVES

JAHARUDDIN

Department of Mathematics,
Faculty of Mathematics and Natural Sciences,
Bogor Agricultural University
Jl. Raya Pajajaran, Kampus IPB Baranangsiang, Bogor, Indonesia

Abstract. By using asymptotic methods, evolution equation is derived for the internal waves in density stratified fluid. This evolution equation arise as a solvability condition. A higher-order extension of the familiar Korteweg-de Vries equation is produced for internal waves in a density stratified flow with a free surface. All coefficients of this extended Korteweg-de Vries equation are expressed via integrals of the modal function for the linear theory of long internal waves.

Key words: stratified fluid, KdV equation, asymptotic expansion

1. Introduction

The Korteweg-de Vries (KdV) equation is a well-known model for the description of nonlinear long internal waves in a fluid stratified by density. The steady-state version of this equation was produced by Long (1953), while Benney (1966) gave the integral expressions for calculation of the coefficients of the Korteweg-de Vries equation for waves in a fluid with arbitrary stratification in the density and current. The next step was due to Lee and Beardsley (1974) who indicated the asymptotic procedure needed to produce higher-order Korteweg-de Vries equations based on two small parameters representing dispersion and nonlinearity. More detailed information was obtained for interfacial waves in a two-layer fluid, and in particular, Kakutani and Yamasaki (1978) found the coefficient of the cubic nonlinear term in an implicit form, and showed its importance for the certain conditions (i.e. the pycnocline lies in the middle of the fluid), in which case the quadratic and cubic nonlinear terms are of the same order. Due to the negative sign of the coefficient of the cubic nonlinear term, this situation leads to an upper limit for the solitary wave amplitude. Then all nonlinear dispersive coefficients for all second order terms were found for a two-layer fluid (Koop and Butler, 1981), and the extended Korteweg-de Vries equation was compared with results of laboratory experiments of internal solitary waves. The goal of this paper is to obtain a higher-order Korteweg-de Vries equation for the internal waves in arbitrary density and current stratified fluid without using the Boussinesq approximation, and also taking into account the free surface.

2. Governing equations

We consider two-dimensional motions of an ideal incompressible fluid which is bounded above by a free surface and below by a rigid boundary h. The governing equations of such fluid are shown below:

$$\begin{aligned} \rho_t + u\rho_x + w\rho_z &= 0 \\ u_x + w_z &= 0 \\ \rho(u_t + uu_x + wu_z) + p_x &= 0 \\ \rho(w_t + uw_x + ww_z) + p_z + \rho g &= 0, \end{aligned} \tag{2.1}$$

where x, z are the horizontal and vertical coordinates, respectively, u, w are the horizontal and vertical velocities, respectively, p denotes the pressure, g is the gravitational acceleration, and ρ is the density. The boundary conditions are

$$\begin{aligned} w &= 0 \qquad \text{at} \qquad z = -h \\ p &= 0 \qquad \text{at} \qquad z = \eta(x,t) \\ \eta_t + u\eta_x &= w \qquad \text{at} \qquad z = \eta(x,t). \end{aligned} \tag{2.2}$$

Here, the fluid has undisturbed constant depth h, and η is the displacement of the free surface from its undisturbed position $z = 0$.

First we introduce $\zeta(x,z,t)$ as the vertical displacement of a fluid particle from its undisturbed position, such that $w = \frac{D\zeta}{Dt}$, where $\frac{D}{Dt} = \partial_t + u\partial_x + w\partial_z$ is the convective time derivative. We suppose that the density of the fluid in the rest state is given by $\rho_o(z)$. Then the density of the fluid in the disturbed state is $\rho(x,z,t) = \rho_o(z - \zeta(x,z,t))$. So that the first equation of (2.1) is now satisfied. Also, it is convenient to express the pressure in the form,

$$p(x,z,t) = -\int_0^z g\rho_o(z')dz' + q(x,z,t).$$

The isopycnal surfaces (i.e. $\rho(x,z,t)$= constant) are then given by

$$z = Z + \zeta(x,z,t). \tag{2.3}$$

where Z is the level as $x \to \pm\infty$. In particular, we let $\zeta(x,z,t) = \xi(x,Z,t)$. In terms of ξ, the kinematic boundary condition (the third equation of (2.2)) becomes simply $\xi = \eta$ at $z = \eta(x,t)$. Next, we determine how the equations transform, when we change the (x,z,t) coordinate to (x,Z,t). We have relations $u(x,z,t) = U(x,Z,t)$, $w(x,z,t) =$

$W(x,Z,t)$, and $q(x,z,t)=Q(x,Z,t)$.

Then we can rewrite equations (2.1) as below:

$$\begin{aligned} U_x + W_Z - \frac{1}{1+\xi_Z}(U_Z\xi_x + W_Z\xi_Z) &= 0 \\ \rho_o(Z)(U_t+UU_x) + Q_x - \frac{1}{1+\xi_Z}Q_Z\xi_x &= 0 \quad (2.4) \\ \rho_o(Z)(W_t+UW_x) + \frac{1}{1+\xi_Z}Q_Z + g(\rho_o(Z)-\rho_o(Z+\xi)) &= 0, \end{aligned}$$

and the boundary conditions (2.2) becomes

$$\begin{aligned} \xi &= 0 && \text{at} \quad Z=-h \\ \int_0^{\xi} g\rho_o(z)dz &= Q(x,Z,t) && \text{at} \quad Z=0. \end{aligned}$$

Also, the vertical velocity becomes

$$W = \xi_t + U\xi_x. \tag{2.5}$$

Using (2.5), we can express (2.4) in term of $U(x,Z,t)$, and $\xi(x,Z,t)$ as below:

$$\begin{aligned} (1+\xi_Z)U_x + (\partial_t + U\partial_x)\,\xi_Z &= 0 \\ (\rho_o(U_t+UU_x))_Z + \xi_x\left(\rho_o(\partial_t+U\partial_x)^2\xi\right)_Z - & \quad (2.6) \\ (1+\xi_Z)\left(\rho_o(\partial_t+U\partial_x)^2\xi\right)_x + g\rho_{oZ}\xi_x &= 0. \end{aligned}$$

The boundary conditions for these two equations are

$$\begin{aligned} \xi &= 0 && \text{at } Z=-h \\ g\xi_x &= -(U_t+UU_x) - \xi_x\,(\partial_t+U\partial_x)^2\,\xi && \text{at } Z=0. \quad (2.7) \end{aligned}$$

Thus, the governing equations are equations (2.6) with boundary conditions at the surface and bottom (2.7). We will use these to obtain higher order Korteweg - de Vries equation for internal waves.

3. Asymptotic expansion

We suppose that the waves are long but finite, and their amplitude is small. We introduce the small parameter ϵ to describe long waves, and hence define the slow variables $X=\epsilon x$, $T=\epsilon t$. Then we let the nonlinear parameter be α, and anticipate the KdV scaling $\alpha=\epsilon^2$. We introduce the scaled variables

$$\theta = X - cT, \;\; \tau = \alpha T \tag{3.1}$$

where c is the speed of a linear long wave (yet to be determined). Then on substituting (3.1) into (2.6), we find that

$$\begin{aligned} (c\rho_o U_\theta)_Z - g\rho_{oZ}\xi_\theta &= F_1 \\ U_\theta - c\xi_{\theta Z} &= F_2, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} F_1 &= -(\rho_o(\alpha U_\tau + UU_\theta))_Z + \epsilon^2(1+\xi_Z)(\rho_o F_3)_\theta - \epsilon^2 \xi_\theta(\rho_o F_3)_Z \\ F_2 &= -\alpha \xi_{Z\tau} - (U\xi_Z)_\theta \\ F_3 &= ((U-c)\partial_\theta + \alpha\partial_\tau)^2 \xi, \end{aligned}$$

and boundary conditions (2.7) become

$$\begin{aligned} \xi &= 0, & \text{at} \quad Z &= -h \\ g\xi_\theta - cU_\theta &= -(\alpha U_\tau + UU_\theta + \epsilon^2 \xi_\theta F_3), & \text{at} \quad Z &= 0. \end{aligned} \tag{3.3}$$

Here the left-hand side of these equations, when equated to zero, describe the linear long wave theory, and thus form the basis of our asymptotic expansion. Equations (3.2) can be reduced to one equation containing ξ only

$$\left(c^2\rho_o\xi_{\theta Z}\right)_Z + \rho_o N^2 \xi_\theta = G, \tag{3.4}$$

where $G = -\left(c\rho_o F_2\right)_Z - F_1$, dan $N^2(Z) = -\frac{g\rho_{oZ}}{\rho_o}$. The boundary conditions (3.3) become

$$\begin{aligned} \xi &= 0, & \text{at} \quad Z &= -h \\ g\xi_\theta &= c^2\xi_{\theta Z} + cF_2 + F_4, & \text{at} \quad Z &= 0, \end{aligned} \tag{3.5}$$

where $F_4 = -(\alpha U_\tau + UU_\theta + \epsilon^2\xi_\theta F_3)$.

We assume that our internal wave field (i.e. the vertical displacement and the horizontal component of velocity) has the asymptotic expansion

$$\begin{aligned} \xi &= \alpha A(\theta,\tau)\phi(Z) + \alpha^2 \xi_2 + \cdots \\ U &= U_o(Z) + \alpha U_1 + \alpha^2 U_2 + \cdots, \end{aligned} \tag{3.6}$$

After substitution of (3.6) into equation (3.4) and the boundary conditions (3.5), and collecting terms of the same order in α, we obtain at the lowest order the equation determining the function $\phi(Z)$, and the speed c,

$$\begin{aligned} \left(\rho_o(U_o-c)^2\phi_Z\right)_Z + \rho_o N^2\phi &= 0 & -h < Z &< 0 \\ \phi &= 0 & \text{at} \quad Z &= -h \\ (U_o-c)^2\phi_Z - g\phi &= 0 & \text{at} \quad Z &= 0. \end{aligned} \tag{3.7}$$

Note that since the differential equation for $\phi(Z)$ is homogenous, we are free to impose a normalization condition on $\phi(Z)$. A commonly used condition is that $\phi(Z_m) = 1$, where $\mid \phi(Z) \mid$ achieves a maximum value at $Z = Z_m$. In this case the amplitude αA is uniquely defined as the amplitude of ξ (to $\mathcal{O}(\alpha)$) at the depth Z_m. Then, at the next order, we obtain the equation for ξ_2,

$$\begin{aligned} \left(\rho_o(U_o-c)^2\xi_{2\theta Z}\right)_Z + \rho_o N^2\xi_{2\theta} &= F & -h < Z &< 0 \\ \xi_{2\theta} &= 0 & \text{at} \quad Z &= -h \\ (U_o-c)^2\xi_{2\theta Z} - g\xi_{2\theta} &= M & \text{at} \quad Z &= 0, \end{aligned} \tag{3.8}$$

Here the inhomogeneous terms F, M are known in terms of $A(\theta,\tau)$ and $\phi(z)$, and are given by

$$\begin{aligned} F &= -2\left(\rho_o(U_o-c)\phi_Z\right)_Z A_\tau + 3\left(\rho_o(U_o-c)^2\phi_Z^2\right)_Z AA_\theta \\ &\quad -\rho_o(U_o-c)^2\phi A_{\theta\theta\theta}, \\ M &= -2(U_o-c)\phi_Z A_\tau + 3(U_o-c)^2\phi_Z^2 AA_\theta. \end{aligned} \tag{3.9}$$

Note that the left-hand side of the equation (3.8) is identical to the equations defining the function ϕ (i.e. (3.7)), and hence can be solved only if a certain solvability condition is satisfied. The solvability condition, that is, the condition for solvability of the inhomogeneous problem (3.8) is

$$\int_{-h}^{0} F\phi dZ = \rho_o(U_o-c)^2(\xi_{2\theta}\phi_Z - \xi_{2\theta Z}\phi)\mid_{Z=-h}^{Z=0}. \tag{3.10}$$

Substituting the expression (3.9) into (3.10) we obtain the required evolution equation for A,

$$A_\tau + \mu AA_\theta + \delta A_{\theta\theta\theta} = 0. \tag{3.11}$$

Here, the coefficients μ and δ are given by

$$\mu = \frac{3\int_{-h}^{0}\rho_o(U_o-c)^2\phi_Z^3 dZ}{2\int_{-h}^{0}\rho_o(c-U_o)\phi_Z^2 dZ},\ \delta = \frac{\int_{-h}^{0}\rho_o(U_o-c)^2\phi^2 dZ}{2\int_{-h}^{0}\rho_o(c-U_o)\phi_Z^2 dZ}. \tag{3.12}$$

Equation (3.11) is the well-known Korteweg-de Vries equation for internal waves.

4. Higher order Korteweg-de Vries equation

Substituting A_τ in (3.11) into (3.9) we obtain

$$\begin{aligned} F &= \left(2\delta\left(\rho_o(U_o-c)\phi_Z\right)_Z - \rho_o(U_o-c)^2\phi\right) A_{\theta\theta\theta} + \\ &\quad \left(3\left(\rho_o(U_o-c)^2\phi_Z^2\right)_Z + 2\mu\left(\rho_o(U_o-c)\phi_Z\right)_Z\right) AA_\theta \\ M &= 2\delta(U_o-c)\phi_Z A_{\theta\theta\theta} + \left(2\mu(U_o-c)\phi_Z + 3(U_o-c)^2\phi_Z^2\right) AA_\theta. \end{aligned}$$

The solution of equation (3.8) in homogeneous term is $A_2\phi$, while $T(Z)A_{\theta\theta} + \hat{T}(Z)A^2$ is a inhomogeneous solution of (3.8). So the general solution of the boundary value problem (3.8) is

$$\xi_2 = A_2(\theta,\tau)\phi(Z) + T(Z)A_{\theta\theta} + \hat{T}(Z)A^2 \tag{4.1}$$

where $T(Z)$ is the first nonlinear correction to the modal structure of internal wave; it is solution of

$$\begin{aligned} (\rho_o \quad &- \quad (U_o\, c)^2 T_Z\big)_Z + \rho_o N^2 T = 2\delta\left(\rho_o(U_o-c)\phi_Z\right)_Z - \rho_o(U_o-c)^2\phi, \\ & \qquad\qquad -h < Z < 0 \\ T &= 0 \qquad\qquad \text{at} \quad Z=-h, \\ gT &= (U_o-c)^2 T_Z - 2\delta(U_o-c)\phi_Z \quad \text{at} \quad Z=0 \end{aligned} \tag{4.2}$$

while $\hat{T}(Z)$ is the first dispersion correction to the modal structure of internal wave; it is solution of

$$\begin{aligned}
\left(\rho_o(U_o - c)^2\hat{T}_Z\right)_Z + \rho_o N^2\hat{T} &= 2\mu\left(\rho_o(U_o-c)\phi_Z\right)_Z \\
&\quad + \frac{3}{2}\left(\rho_o(U_o-c)^2\phi_Z^2\right)_Z, \ -h < Z < 0 \\
\hat{T} &= 0 \qquad \text{at } Z = -h, \qquad (4.3)\\
g\hat{T} &= (U_o-c)^2\hat{T}_Z - \mu(U_o-c)\phi_Z \\
&\quad -\frac{3}{2}(U_o-c)^2\phi_Z^2 \qquad \text{at } Z = 0.
\end{aligned}$$

It is important to note that solutions of the boundary-value problems (4.2) and (4.3) are unique only up to additive multiples of ϕ. This problem was discussed in Lamb and Yan (1996), and Holloway and Pelinovsky (2001). It is convenient to let $A_2(\theta,\tau)$ represent the isopycnal displacement at the level $Z = Z_m$ where there is a maximum in the linear mode $\phi(Z)$. Hence we choose the auxiliary conditions $T(Z_m) = 0$ and $\hat{T}(Z_m) = 0$. In this case the series (3.6), using (4.1), at the point $Z = Z_m$ is

$$\alpha(A + \alpha A_2).$$

After substitution of (3.6) into equation (3.4) and the boundary conditions (3.5), and collecting terms of the same order in α, we obtain

$$\begin{aligned}
U_1 &= -(U_o-c)\phi_Z A \\
U_2 &= -(U_o-c)\phi_Z A_2 + \left(\delta\phi_Z - (U_o-c)T_Z +\right) A_{\theta\theta} + \left((U_o - c)\phi_Z^2\right. \\
&\quad \left. + \frac{1}{2}\,\mu\phi_Z - (U_o-c)\hat{T}_Z\right) A^2.
\end{aligned}$$

Then, at the next order ($\mathcal{O}(\alpha^3)$), we obtain the equation for ξ_3,

$$\begin{aligned}
\left(\rho_o(U_o-c)^2\xi_{3\theta Z}\right)_Z + \rho_o N^2\xi_{3\theta} &= \hat{F} \quad -h < Z < 0, \\
\xi_{3\theta} &= 0 \text{ at } \quad Z = -h, \qquad (4.4)\\
(U_o-c)^2\xi_{3\theta Z} - g\xi_{3\theta} &= \hat{M} \quad \text{at} \quad Z = 0.
\end{aligned}$$

Here the inhomogeneous terms $\hat{F}$, $\hat{M}$ are given by

$$\begin{aligned}
\hat{F} &= b_1 A_{2\tau} + b_2(AA_2)_\theta + b_3 A_{2\theta\theta\theta} + b_4 A_{\theta\theta\theta\theta\theta} + b_5 A^2 A_\theta \\
&\quad + b_6 AA_{\theta\theta\theta} + b_7 A_\theta A_{\theta\theta} \qquad (4.5)\\
\hat{M} &= a_1 A_{2\tau} + a_2(AA_2)_\theta + a_3 A_{\theta\theta\theta\theta\theta} + a_4 A^2 A_\theta + a_5 AA_{\theta\theta\theta} + a_6 A_\theta A_{\theta\theta}.
\end{aligned}$$

The coefficients a_i and b_i are given by

$$
\begin{aligned}
a_1 &= -2(U_o - c)\phi_Z \\
a_2 &= 3(U_o - c)^2\phi_Z^2 \\
a_3 &= 2\delta(U_o - c)T_Z - \delta^2\phi_Z \\
a_4 &= 4\mu(U_o - c)\hat{T}_Z - 6(U_o - c)^2\phi_Z^3 + 9(U_o - c)^2\phi_Z\hat{T}_Z - 5\mu(U_o - c)\phi_Z^2 - \mu^2\phi_Z \\
a_5 &= 2\mu(U_o - c)T_Z + 4\delta(U_o - c)\hat{T}_Z + 3(U_o - c)^2\phi_Z T_Z - 4\delta(U_o - c)\phi_Z^2 - 2\mu\delta\phi_Z \\
a_6 &= 6\mu(U_o - c)T_Z + 3(U_o - c)^2\phi_Z T_Z - 2\delta(U_o - c)\phi_Z^2 - (U_o - c)^2\phi^2 - 3\mu\delta\phi_Z \\
b_1 &= (\rho_o a_1)_Z \\
b_2 &= (\rho_o a_2)_Z \\
b_3 &= (\rho_o(U_o - c)^2\phi \\
b_4 &= (\rho_o a_3)_Z - 2\delta\rho_o(U_o - c)\phi + \rho_o(U_o - c)^2 T \\
b_5 &= (\rho_o a_4)_Z \\
b_6 &= (\rho_o a_5)_Z - 2\mu\rho_o(U_o - c)\phi - \rho_o(U_o - c)^2\phi\phi_Z + 2\rho_o(U_o - c)^2\hat{T} \\
b_7 &= (\rho_o a_6)_Z - 6\mu\rho_o(U_o - c)\phi - 3\rho_o(U_o - c)^2\phi\phi_Z + 6\rho_o(U_o - c)^2\hat{T}.
\end{aligned}
$$

The condition for solvability of the inhomogeneous problem (4.4) is

$$\int_{-h}^{0} \hat{F}\phi dZ = \rho_o(U_o - c)^2(\xi_{3\theta Z}\phi - \xi_{3\theta}\phi_Z)\mid_{Z=-h}^{Z=0} . \tag{4.6}$$

Substituting the expression (4.5) into (4.6), we obtain the required evolution equation of A_2,

$$A_{2\tau} + \mu(AA_2)_\theta + \delta A_{2\theta\theta\theta} = \beta_1 A_{\theta\theta\theta\theta\theta} + \beta_2 A^2 A_\theta + \beta_3 AA_{\theta\theta\theta} + \beta_4 A_\theta A_{\theta\theta}. \tag{4.7}$$

Here, the coefficients μ and δ are given by (3.12), while β_1, β_2, β_3, and β_4 are given by

$$
\begin{aligned}
\beta_1 &= \frac{1}{I}\int_{-h}^{0} \rho_o\left(-(U_o - c)^2\phi T + \delta^2\phi_Z^2 + 2\delta(U_o - c)(\phi^2 - \phi_Z T_Z)\right) dZ \\
\beta_2 &= \frac{1}{I}\int_{-h}^{0} \rho_o\left(3(U_o - c)^2\phi_Z^2(2\phi_Z^2 - 3\hat{T}_Z) + \mu^2\phi_Z^2 + \mu(U_o - c)(5\phi_Z^2 - 4\hat{T}_Z)\phi_Z\right) dZ \\
\beta_3 &= \frac{1}{I}\int_{-h}^{0} \rho_o(2\mu\delta\phi_Z^2 + 2\mu(U_o - c)\phi^2 + (U_o - c)^2\phi^2\phi_Z - (U_o - c)^2(2\phi\hat{T} + 3\phi_Z^2 T_Z) \\
&\quad -2(U_o - c)(\mu T_Z + 2\delta\hat{T}_Z)\phi_Z + 4\delta(U_o - c)\phi_Z^3)dZ \\
\beta_4 &= \frac{1}{I}\int_{-h}^{0} \rho_o((U_o - c)(2\delta\phi_Z^3 + 6\mu\phi^2) + 3\mu\delta\phi_Z^2 + 2(U_o - c)^2(\phi^2\phi_Z - 3\phi\hat{T}) \\
&\quad -6\mu(U_o - c)\phi_Z T_Z - 3(U_o - c)^2\phi_Z^2 T_Z)dZ \\
I &= -2\int_{-h}^{0} \rho_o(U_o - c)\phi_Z^2 dZ.
\end{aligned}
$$

The isopycnal displacement at the level $Z = Z_m$ in third order $\mathcal{O}(\alpha^3)$ is

$$\begin{aligned}\xi &= \alpha A(\theta,\tau) + \alpha^2 A_2(\theta,\tau) + \mathcal{O}(\alpha^3)\\ &= \alpha B(\theta,\tau) + \mathcal{O}(\alpha^3)\end{aligned}$$

where $B = A + \alpha A_2$. Then, one again using (4.7), and the KdV equation (4.32), and neglecting terms of $\mathcal{O}(\alpha^3)$, we obtain the higher-order KdV equation,

$$\begin{aligned}B_\tau &+ \mu B B_\theta + \delta B_{\theta\theta\theta} +\\ &\alpha\left(\beta_1 B_{\theta\theta\theta\theta\theta} + \beta_2 B^2 B_\theta + \beta_3 B B_{\theta\theta\theta} + \beta_4 B_\theta B_{\theta\theta}\right) = 0. \quad (4.8)\end{aligned}$$

However, we must now point out that this higher order Korteweg-de Vries equation (4.8) is an asymptotic result valid when α is sufficiently small, and is must likely to be useful when the coefficient μ of the quadratic nonlinear term is small. Nevertheless, because observed internal waves are often quite large, it may be useful to use (4.8) as the model equation even when μ is not small.

5. Conclusion

We have presented an evolution equation, the extended Korteweg-de Vries equation, to describe internal waves in an arbitrary density and current stratified flow, without using the Boussinesq approximation and with a free surface, valid to the second order of perturbation theory. All the coefficients of this equation are given explicitly as integrals of the modal function and its nonlinear and dispersion correction. It is important to note that our derivation is completely general.

References

[1] Benney, D.J. (1966), Long nonlinear waves in fluid flows, *J. Math. Phys.*, **45**, 52-63

[2] Holloway, P., E. Pelinovsky (2001), Internal tide transformation and oceanic internal solitary waves, dalam Stratified Flows, Bab 2, Grimshaw, R., Editor, Kluwer, Dordrecht

[3] Kakutani, T. and Yamasaki, N. (1978), Solitary waves on a two-layer fluid, *J. Phys. Soc. Japan*, **45**, 674-679

[4] Koop, C.G. and Butler G. (1981), An investigation of internal solitary waves in a two fluid system, *J. Fluid Mech.*, **112**, 225-251

[5] Long, R.R., 1953. *Some Aspects of the Flow of Stratified Fluids*, Tellus, 5, 42-58.

[6] Lee, C.Y., R.C. Beardsley (1974), The generation of long nonlinear internal waves in weakly stratified shear flow, *J. Geophys. Res.*, **79**, 453-462

[7] Lamb, K., L. Yan (1996), The evolution of internal wave undular bores: comparisons of a fully nonlinear numerical model with weakly nonlinear theory, *J. Phys. Oceanography*, **26**, 2712-2734